Math 829: Algebraic Topology
Homework 2

Due Wednesday, February 28th

1. Let $\mathbb{F}$ a field. Define[1]
$$\text{Free} : \underline{\textsc{Set}} \to \underline{\textsc{Vect}/\mathbb{F}}$$
to take a set S to the $\mathbb{F}$-vector space with basis S. Explicitly, $\text{Free}(S)$ is the collection of formal $\mathbb{F}$-linear combinations of elements of S, with addition and scalar multiplication inherited from $\mathbb{F}$. Show that Free is a functor. Conclude that if two vector spaces have bases of the same cardinality, they are isomorphic.

2. Let $\mathbb{F}$ a field. Define
$$\text{Forget} : \underline{\textsc{Vect}/\mathbb{F}} \to \underline{\textsc{Set}}$$
to be the *forgetful* mapping taking a $\mathbb{F}$-vector space V to its underlying set of vectors. Show that Forget is a functor.

3. Let S be a set and W a $\mathbb{F}$-vector space. Show that every function $S \to \text{Forget}(W)$ extends uniquely to a linear transformation $\text{Free}(S) \to W$; and, conversely, that every linear transformation $Free(S) \to W$ restricts uniquely to a function $S \to \text{Forget}(W)$. That is,
$$Hom_{\underline{\textsc{Vect}/\mathbb{F}}}(\text{Free}(S), W) \cong Hom_{\underline{\textsc{Set}}}(S, \text{Forget}(W)).$$
This is called an *adjunction* between Free and Forget, and is the categorical statement of the fact that every linear transformation between vector spaces (allegedly a complicated thing) is uniquely determined by the image of the basis vectors (a much simpler thing).

4. Hatcher, Sec 1.1, Exercise 3.

5. Hatcher, Sec 1.1, Exercise 6.

6. Hatcher, Sec 1.1, Exercise 10.

7. Hatcher, Sec 1.1, Exercise 13.

[1] Yes, that's intentionally the same name as in the preliminary exercise. Think about what the analog of exercises 1-3 looks like in the world of free groups.