Math 456: Topology and Its Applications
Homework 6
Due Friday, October 20th

1. Prove the *Five lemma*. For inspiration, look at the proof of the snake lemma.

   Lemma (Five lemma)**.** *Consider the commutative diagram*

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{f_1} & V_2 & \xrightarrow{f_2} & V_3 & \xrightarrow{f_3} & V_4 & \xrightarrow{f_4} & V_5 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ W_1 & \xrightarrow{g_1} & W_2 & \xrightarrow{g_2} & W_3 & \xrightarrow{g_3} & W_4 & \xrightarrow{g_4} & W_5 \end{array}$$

   Suppose that the rows are exact at the second, third and fourth entries, and that h_1, h_2, h_4 and h_5 are isomorphisms. Then h_3 is an isormorphism.

2. Let S^n be the usual sphere with equator S^{n-1}. Compute the homology of the quotient space S^n/S^{n-1}. Draw this space when $n = 1$ and $n = 2$.

3. Prove the Perron-Frobenius theorem.

   Theorem 1 (Perron-Frobenius)**.** *Let M be an $n \times n$ matrix with all positive entries. Then M has a positive eigenvalue λ and associated eigenvector $\vec{v}$ which has all positive entries.*

   Hint: what is the image under M of $\mathbb{R}^n_{\geq 0}$?

4. Prove the intermediate value theorem.

   Theorem 2 (IVT)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) = c$ and $f(b) = d$, then $[c, d] \subseteq im(f)$.*

   Prove by contradiction, using the long exact sequence of a pair.