Math 456: Topology and Its Applications
Homework 5
Due Friday, October 6th

1. Prove the following lemma from the lecture notes.

   Lemma. *Let $T : V \to W$ be a linear transformation, then $ker(T)$ is a subspace of V and $im(T)$ is a subspace of W.*

2. Prove the following lemma from the lecture notes.

   Lemma. *Let Σ be a simplicial complex with chain complex $C_\bullet(\Sigma)$. For any k, $d_{k-1} \circ d_k = 0$.*

3. Prove the so-called "First Isomorphism Theorem".

   Theorem 1. *Let $T : V \to W$ be a linear transformation. Then $V/ker(T) \cong im(T)$.*

   Please write a direct proof – don't appeal to the Rank-Nullity theorem. Instead, apply the same sort of tools used in the proof of the Rank-Nullity theorem.

4. Let

   - $\Sigma_1 = (\{1, 2, 3, 4, 5\}, \{12, 14, 23, 34, 235\})$,
   - $\Sigma_2 = (\{1, 2, 3, 4\}, \{12, 13, 14, 23, 24, 34\})$,
   - $\Sigma_3 = (\{1, 2, 3, 4\}, \{123, 124, 134, 234\})$ (c.f. Homework 4, Problem 3),
   - $\Sigma_4 = (\{1, 2, 3, 4\}, \{1234\})$, and
   - $\Sigma_5 = (\{1, 2, 3, 4, 5, 6\}, \{123, 125, 134, 145, 236, 256, 346, 456\})$. (from Homework 4, Problem 4)

   For each i,

   (a) draw a geometric realization of Σ,
   (b) write down the chain complex $C_\bullet(\Sigma_i)$,
   (c) compute a basis for the kernel and image of each differential in the chain complex $C_\bullet(\Sigma_i)$ (please do this by hand for these small examples – soon, we'll move to big ones and start letting computers run the show),
   (d) compute $H_*(\Sigma_i)$ and $\beta_*(\Sigma_i)$, and
   (e) write down and draw a representative for each non-trivial homology class, and count the number of elements in each such class.

5. Let $X = S^1 \vee S^1$ (see Homework 3, Problem 7). Draw a good open cover $\mathcal{U}$ for X and compute $H_*(N(\mathcal{U}))$. Draw representatives for each non-trivial homology class /on pictures of $S^1 \vee S^1$.

6. Use the Δ-complex structure you created for the torus in Homework 4, Problem 6 to compute $H_*(T^2)$. (Details on computing homology for Δ complexes are available in the notes.) Draw representatives for each non-trivial homology class on a picture of T^2.