Math 456: Topology and Its Applications
Homework 3
Due Friday, September 22nd

1. Let $X = \{p, q\}$. Classify all continuous functions $f : \mathbb{R} \to X$ and $g : X \to \mathbb{R}$ when τ_X is
   (a) the trivial topology, $\{\varnothing, X\}$,
   (b) the Sierpenski topology, $\{\varnothing, \{p\}, X\}$, and
   (c) the discrete topology, $\{\varnothing, \{p\}, \{q\}, X\}$.

2. (a) Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Describe the types of sets that can occur as the image of an open interval under f and contras to those that occur as the preimage of an open interval.

   (b) Prove the following theorem from the lecture notes.

   Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then*

   $$\text{for each } x \in \mathbb{R}, \text{ for each } \epsilon_x > 0 \text{ there exists a } \delta_x > 0 \text{ so that if } |x - y| < \delta_x \text{ then } |f(x) - f(y)| < \epsilon_x$$
   $$\Longleftrightarrow$$
   $$\text{for every open set } U \subseteq \mathbb{R},\ f^{-1}(U) \text{ is open.}$$

   That is, the two definitions of continuity agree on $\mathbb{R}$.

   (c) Prove the following lemma.

   Lemma. *Let X, Y, Z be topological spaces and $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then the composite $(g \circ f) : X \to Z$ is continuous.*

   Compare your proof to the "calculus proof" using ϵ-δ definitions for evidence that abstraction to the appropriate level is our friend. (Seriously – go find it in your notes or online!)

3. Let $\Gamma = (\{v_0, v_1, v_2, v_3, v_4\}, \{v_9v_1, v_0v_2, v_0v_3, v_2v_4, v_3v_4\})$. Draw a realization of Γ in $\mathbb{R}^2$ and use it to illustrate the kinds of open sets that appear in the topological realization of Γ.

4. Consider the quotient space $X = D^2/\sim$ where the equivalence relation identifes all points in the image of the inclusion $\iota : S^1 \hookrightarrow D^2$ to a single point. What do open sets in X look like? (Draw representative examples on a picture of the disk.) What familiar topological space is this, really? Draw a picture that illustrates how open sets in the familiar space correspond to those in X. Does this generalize?

5. Draw the Reeb graph for the saddle surface X in $\mathbb{R}^3$ given by

   $$X = \{(x, y, z) \mid z = x^2 - y^2, x, y \in [-1, 1]\}$$

   using $f : X \to \mathbb{R}$ given by $f(x, y, z) = z$. Nicely draw or use a computer to plot the surface and visually show how the the graph relates to the surface.

6. Prove that homotopy equivalence is an equivalence relation.

7. Let $S^1 \vee S^1 = (S^1_a \sqcup S^1_b / \sim)$ where the equivalence relation is given by $(1,0)_a \sim (-1,0)_b$.

   (a) Use an illustration to demonstrate that $(\mathbb{R}^2 \setminus \{(-1,0),(1,0)\}) \simeq S^1 \vee S^1$.

   (b) Draw a sequence of pictures illustrating that deleting a single point $\{p\}$ from the torus T^2 makes $T^2 \setminus \{p\} \simeq S^1 \vee S^1$.

   Using the previous problem, we see that one hole in a torus is the same as two holes in a plane.

8. If a map $f : X \to Y$ is homotopic to a constant map ($C(x) = c$ for some $c \in Y$), we say f is *null-homotopic*. Show that

   (a) $f : S^1 \to \mathbb{R}^3 \setminus \{0\}$ by $(x,y) \mapsto (x,y,0)$, and

   (b) $g : S^1 \to \mathbb{R}^2 \setminus \{0\}$ by $(x,y) \mapsto (x+2,y)$

   are null-homotopic maps.