Math 456: Topology and Its Applications
Homework 2

Due Friday, September 15th

1. Prove the following lemma from the lecture notes.

   Lemma. *Let Γ be a tree. Any two vertices in Γ are connected by a unique simple path.*

2. (a) Derive the formula for stereographic projection from the unit sphere $S^2 \subset \mathbb{R}^3$ to the (x, y)-plane using the point $P = (0, 0, 1)$. (Hint: parameterize the line.)

   (b) What is the image in the plane of the upper hemisphere? The lower hemisphere?

   (c) What is the inverse image in the sphere of a line in the plane? Of a circle in the plane?

3. Prove the following lemma from the lecture notes. (The notation in the notes for spanning trees and spherical duals is far better than what I used in the lecture, so let's use that notation.)

   Lemma. *Let Γ be a planar graph with realization ρ in S^2 and spanning tree $\tau(\Gamma)$. The spherical dual $\tau^*(\Gamma, \rho)$ is a tree.*

4. (a) Draw a planar graph with six vertices that maximizes the number of edges a planar graph with six vertices can have.

   (b) Prove the following corollary from the lecture notes.

   Corollary 12. *Let Γ be a connected planar graph with $|V| \geq 3$. Then $|E| \leq 3|V| - 6$.*

5. The *discrete metric* on a set X is given by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. Prove that the discrete topology is the topology obtained by taking open sets to be unions of ϵ-balls around points in x.

6. Take $X = \mathbb{R}$ and τ generated by all half-open intervbals $[a, b)$ with $a, b \in \mathbb{R}$. Is $(a, b]$ open or closed in this topology?