APPLIED TOPOLOGY LECTURE NOTES

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Homological Algebra

Now that we have a firm grasp on the objects we're going to be studying, we need a quantitative framework we can use to do the computations we will use to explore properties of those objects. As is nearly always the case in mathematics, the best tool for the job is algebra, in this case a sub-field called Homological Algebra. Here, we will focus on a version built out of vector spaces so we can rely on our understanding of linear algebra. Because computers are very good at linear algebra, this is the principal approach used in applications.

Application: Nash equilibria

Let's close this discussion with a pair of applications of the computation we just completed.

Theorem 1 (Brouwer fixed point theorem)**.** *Let $f : D^n \to D^n$ be a continuous function. Then f has a fixed point; e.g. there is $x \in D^n$ with $f(x) = x$.*

That is, when you stir your coffee, there's always a point on the surface of the liquid that's doesn't move (which point, exactly, varies continuously with the Δt between measurements).

Proof. Suppose the converse, that f has no fixed point. Then we can use the function f to build another function $r : D^n \to S^{n-1}$ by taking $r(x)$ to be the point on the ray from $f(x)$ to x which intersects $S^{n-1} \subseteq D^n$. Because there are no fixed points, this ray is always well defined, and varying x (and thus $f(x)$ a small amount similarly varys the ray only slightly, so the map is continuous.

Further, if $x \in S^{n-1}$, then $r(x) = x - r(x)$ restricts to $\mathrm{id}_{S^{n-1}}$ on the boundary. In particular, writing ι for the subspace inclusion, $r \circ \iota = \mathrm{id}_{S^{n-1}} : S^{n-1} \to S^{n-1}$ is the identity map. Thus, the induced map $((r \circ \iota)_*)_{n-1} : H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$ must equal the identity map $((\mathrm{id}_{S^{n-1}})_*)_{n-1} : H_{n-1}(S^{n-1}) = \mathbb{F}_2 \to H_{n-1}(S^{n-1}) = \mathbb{F}_2$. But $(\iota_{ast})_{n-1} : H_{n-1}(S^{n-1}) = \mathbb{F}_2 \to H_{n-1}(D^n) = 0$ must be the zero map, and thus so must the composite $((r \circ \iota)_*)_{n-1}$. Thus, no such map f can exist. $\square$

We should finally stop and define a couple of things. Let X be a topological space and $Y \subseteq X$ a subspace. The *boundary* of Y in X, written $\partial_X Y$ (or just ∂Y when X is clear) is the set of all points $y \in X$ so that every open set U with $y \in U$ has both $U \cap Y \neq \varnothing$ and $U \setminus Y \neq \varnothing$ – that is, the boundary is all points so that every open neighborhood of the point contains points outside the set. The *interior* of Y is $\mathrm{int}(Y) = Y \setminus \partial Y$, and the *closure* of Y is $\mathrm{cl}(Y) = Y \cup \partial Y$. A set Y is closed if $Y = \mathrm{cl}(Y)$.

We want a slightly more general version of the Brouwer fixed point theorem.

Corollary 2. *Suppose U is a closed, bounded, convex subset of $\mathbb{R}^n$, then any map $f : U \to U$ has a fixed point.*

To prove this theorem, assume $U \subseteq \mathbb{R}^n$ contains the origin, and scale it so it is contained in the unit disk, D^n. Now, observe that "stretching" U along each unit vector provides a continuous, bijective deformation of U to D^n which can be continuously inverted. To find a fixed point, conjugate f with this deformation and its inverse and apply the theorem to D^n. (Observe that we can do this much more generally, whenever we have such a deformation, called a *homeomorphism*.)

The most famous application of the Brower fixed point theorem comes from economics, specifically from game theory, where it led to a Nobel prize.

Definition. An *(n-player finite) game* is a triple $([n], A, u)$ where $[n]$ is the set of *players*, $A = \prod_{k=1}^n A_k$, $|A_k| < \infty$ is the set of *pure strategies*, and $u : A \to \mathbb{R}^n$ is the *utility function*.

Thus, for each $\vec{a} \in A$, a choice of pure strategies for each player, we assign to each player k a utility $u_k(\vec{a})$ based on what everyone chose to do. It is usually more useful to think in terms of probabilities – what is the likelihood that player k employs strategy a_k?

Definition. Let X be a finite set. Write $\mathcal{P}(X)$ for the *set of probability distributions* on X,

$$\mathcal{P}(X) = \{p : X \to \mathbb{R} \mid p(x) \geq 0 \text{ for all } x \in X, \sum_{x \in X} p_i(x) = 1\}.$$

This looks awfully familiar: each p is just a list of non-negative real numbers which sum to 1. In fact, $\mathcal{P}(X) = \Delta^{|X|-1}$.

Definition. The set of *mixed strategies* for a game ([n], A, u) is the set $M = \prod_{k=1}^n M_k$ where $M_k = \mathcal{P}(A_k)$ are the mixed strategies for player k.

Thus, $M = \prod_{k=1}^{n} \Delta_k^{|A_k|-1}$. Observe that this is a convex set of dimension $\sum_{k=1}^{n}(|A_k| - 1)$.

We need a way to measure how these mixed strategies affect utility. To do so, we'll ask what the average outcome for each player is, assuming all players make their strategy choices indepdendently.

Definition. The *expected utility for player k under a mixed strategy* $\vec{p} \in M$ is

$$\bar{u}_k(\vec{p}) = \sum_{\vec{a} \in A} \left(u_k(a) \prod_{k=1}^{n} p_k(a_k) \right).$$

This above expression can be rewritten in terms of the pure strategy choices for player k under uncertainty about the other player's decisions, producing (by slight abuse of notation)

$$\bar{u}_k(\vec{p}) = \sum_{a_k \in A_k} \left(\bar{u}_k(p_1, \dots, p_{k-1}, a_k, p_{k+1}, \dots p_n) p_k(a_k) \right).$$

Definition. Write $\vec{p}_{-k} = (p_1, \dots, \widehat{p}_k, \dots p_n)$. A *best response* for player k to $\vec{p}_{-k}$ is $p_k^* \in M_k$ so that $\bar{u}_k(p_1, \dots, p_k^*, \dots p_n) \geq \bar{u}_k(\vec{p})$ for all $\vec{p} \in M$. A *Nash equilibrium* for a game is a mixed strategy $\vec{p} \in M$ which is a best response to $\vec{p}_{-k}$ for each player $k \in [n]$.

Theorem 3 (Nash, 1951)**.** *Every finite, n-player game has a Nash equilibrium.*

Proof. We begin by defining functions

$$\begin{aligned}
\mathrm{Gain}(\vec{p}, k, a_k) &= \max\left(0, \bar{u}_k(p_1, \dots, p_{k-1}, a_k, p_{k+1}, \dots, p_n) - \bar{u}_k(\vec{p})\right) \\
g(\vec{p}, k, a_k) &= p_k(a_k) + \mathrm{Gain}(\vec{p}, k, a_k) \\
C(\vec{p}, k) &= \sum_{a_k \in A_k} g(\vec{p}, k, a_k) = 1 + \sum_{a_k \in A_k} \mathrm{Gain}(\vec{p}, k, a_k) \\
f(\vec{p}, k, a_k) &= \frac{g(\vec{p}, k, a_k)}{C(\vec{p}, k)}
\end{aligned}$$

which respectively approximate the directional derivative along strategy changes, that derivative translated to the point in question, the gradient and gradient ascent in this context. We think of each of these as a function of $\vec{p}$, so, for example, $f(\vec{p}, -, -) : M \to M$.

Observe that $C \geq 1$

Now, under the discrete dynamics induced by iterating f on M, the existence of fixed points is immediate from Brouwer, since we have already observed that M is a closed, convex, bounded subset of Euclidean space. We need to check that fixed point are Nash Equilibria.

Suppose not: $f(\vec{p}^*, -, -) = \vec{p}^*$, but this is not a best response for some player. Then there are k and a_k so that $\mathrm{Gain}(\vec{p}^*, k, a_k) > 0$, and thus $C(\vec{p}^*, k) > 1$. Now, unpacking f, and solving for the gain at the fixed point, we have

$$\begin{aligned}
\vec{p}^* &= f(\vec{p}^*, -, -) \\
&= \frac{g(\vec{p}^*, -, -)}{C(\vec{p}^*, -)} \\
&= \frac{\vec{p}^* + \mathrm{Gain}(\vec{p}^*, -, -)}{C(\vec{p}^*, -)} \\
\mathrm{Gain}(\vec{p}^*, -, -) &= (C(\vec{p}^*, -) - 1)\vec{p}^*
\end{aligned}$$

Thus, for any k and a_k with non-zero gain at $\vec{p}^*$, the gain is a positive multiple of $\vec{p}^*$.

Finally, we compute how this affects utility

$$\begin{aligned}
0 &= \bar{u}_k(\vec{p}^*) - \bar{u}_k(\vec{p}^*) \\
&= \sum_{a_k \in A_k} \bar{u}_k(p_1^*, \ldots, p_{k-1}^*, a_k, p_{k+1}^*, \ldots p_n^*) p_k^*(a_k) \\
&\quad - \sum_{a_k \in A_k} \bar{u}_k(\vec{p}^*) p_k^*(a_k) \\
&= \sum_{a_k \in A_k} \left(\bar{u}_k(p_1^*, \ldots, p_{k-1}^*, a_k, p_{k+1}^*, \ldots p_n^*) - \bar{u}_k(\vec{p}^*) \right) p_k^*(a_k) \\
&= \sum_{a_k \in A_k} \mathrm{Gain}(\vec{p}^*, k, a_k) p_k^*(a_k) \\
&= \sum_{a_k \in A_k} (C(\vec{p}^*, k) - 1)(p_k^*(a_k))^2 > 0
\end{aligned}$$

Thus, every fixed point must be a Nash Equilibrium. □