APPLIED TOPOLOGY LECTURE NOTES

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Homological Algebra

Now that we have a firm grasp on the objects we're going to be studying, we need a quantitative framework we can use to do the computations we will use to explore properties of those objects. As is nearly always the case in mathematics, the best tool for the job is algebra, in this case a sub-field called Homological Algebra. Here, we will focus on a version built out of vector spaces so we can rely on our understanding of linear algebra. Because computers are very good at linear algebra, this is the principal approach used in applications.

The Snake Lemma

Computing homology directly is a lot of work – the vector spaces get quite large, quite quickly. Fortunately, homology is built in a way that facilitates building complicated computations up from simpler ones. One of the most useful of these is a kind of quotient space construction: given a simplicial complex Σ, select a subcomplex Σ' and define an equivalence relation on $|\Sigma|$ so that $x \sim y$ if $x, y \in |\Sigma'| \subseteq |\Sigma|$.

Definition. Let Σ be a simplicial complex, $\Sigma' \subseteq \Sigma$ a subcomplex with inclusion map ι. The *relative chain complex* $C_\bullet(\Sigma, \Sigma')$ is given by

$$C_k(\Sigma, \Sigma') = C_k(\Sigma) / (\iota_\#)_k(C_k(\Sigma'))$$

with differentials d_k^{rel} induced from d_k by $d_k^{\text{rel}}([e_\sigma]) = [d_k(e_\sigma)]$. The *relative homology* $H_*(\Sigma, \Sigma')$ is the homology of the relative chain complex.

The map $(\iota_\#)_k : C_k(\Sigma') \to C_k(\Sigma)$ simply maps basis vectors for simplices in Σ' to the corresponding basis vectors for simplices in Σ, and is clearly an injective map. There is also a surjective map $\pi_k : C_k(\Sigma) \to C_k(\Sigma, \Sigma')$ which quotients out elements in the image of $(\iota_\#)_k$ – that is, they are zero in the image. Indeed, they are precisely the elements which go to zero, so

$$0 \to C_k(\Sigma') \xrightarrow{(\iota_\#)_k} C_k(\Sigma) \xrightarrow{\pi_k} C_k(\Sigma, \Sigma') \to 0$$

is exact at $C_k(\Sigma)$. It is straightforward to check that this sequence is also exact at $C_k(\Sigma')$ and $C_k(\Sigma, \Sigma')$. A sequence of exactly three vector spaces which is exact at each position is called a *short exact sequence*. Since this is true for all k, and the maps commute with the differentials, we get a short exact sequence of chain complexes

$$0 \to C_\bullet(\Sigma') \xrightarrow{\iota_\#} C_\bullet(\Sigma) \xrightarrow{\pi} C_\bullet(\Sigma, \Sigma') \to 0.$$

Any longer sequence which is exact at every term is called a *long exact sequence*.

Lemma 1 (Snake lemma)**.** *Let*

$$0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0$$

be a short exact sequence of chain complexes. This induces a long exact sequence in homology

$$\begin{array}{l} \cdots \xrightarrow{(g_*)_2} H_2(C''_\bullet) \xrightarrow{\delta_2} \\ H_1(C_\bullet) \xrightarrow{(f_*)_1} H_1(C'_\bullet) \xrightarrow{(g_*)_1} H_1(C''_\bullet) \xrightarrow{\delta_1} \\ H_0(C_\bullet) \xrightarrow{(f_*)_0} H_0(C'_\bullet) \xrightarrow{(g_*)_0} H_0(C''_\bullet) \longrightarrow 0 \end{array}$$

Proof. The maps f_* and g_* are just the usual maps induced by a map on chain complexes. We need to work out what these new δ maps – called "connecting homomorphsims" – are, and show exactness.

Unlike our previous definitions, it's not immediately clear what the connecting map should be. Let's dig in to the object at hand.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{k+1} & \xrightarrow{f_{k+1}} & C'_{k+1} & \xrightarrow{g_{k+1}} & C''_{k+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_{k+1}} & & \downarrow{\scriptstyle d'_{k+1}} & & \downarrow{\scriptstyle d''_{k+1}} & & \\ 0 & \longrightarrow & C_k & \xrightarrow{f_k} & C'_k & \xrightarrow{g_k} & C''_k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_k} & & \downarrow{\scriptstyle d'_k} & & \downarrow{\scriptstyle d''_k} & & \\ 0 & \longrightarrow & C_{k-1} & \xrightarrow{f_{k-1}} & C'_{k-1} & \xrightarrow{g_{k-1}} & C''_{k-1} & \longrightarrow & 0 \end{array}$$

Suppose $\alpha \in \ker(d''_k) \subseteq C''_k$. Since g_k is surjective, there is $\beta \in C'_k$ with $g_k(\beta) = \alpha$. We can push β down through d'_k to get an element $d'_k(\beta) \in C'_{k-1}$. Now, use the fact that g is a map of chain complexes to obtain $g_{k-1}(d'_k(\beta)) = d''_k(g_k(\beta)) = d''_k(\alpha) = 0$. Since this is a short exact sequence, $\ker(g_{k-1}) = \operatorname{im}(f_{k-1})$, so there is $\gamma \in C_{k-1}$ with $f_{k-1}(\gamma) = d'_k(\beta)$. Define $\delta_k([\alpha]) = [\gamma]$.

Now we need to check that this is well-defined. Observe that the only choice we made was the lift under g_k, so suppose there are $\beta \neq \beta' \in C'_k$ with $g_k(\beta) = g_k(\beta') = \alpha$. Follow the process above to get $\gamma, \gamma' \in C_{k-1}$ respectively. Now, $\beta' - \beta \in \ker(g_k)$, so by exactness there is $\sigma \in C_k$ with $f_k(\sigma) = \beta' - \beta$. Apply the injective map f_{k-1} to $\gamma + d_k(\sigma)$, obtaining

$$\begin{aligned} f_{k-1}(\gamma + d_k(\sigma)) &= f_{k-1}(\gamma) + f_{k-1}(d_k(\sigma)) \\ &= d'_k(\beta) + d'_k(f_k(\sigma)) \\ &= d'_k(\beta) + d'_k(\beta' - \beta) \\ &= d'_k(\beta') = f_{k-1}(\gamma'). \end{aligned}$$

But since f_{k+1} is injective, $\gamma' = \gamma + d_k(\sigma)$, so $[\gamma] = [\gamma'] \in H_k(C_\bullet)$.

Checking exactness is left as an exercise. □

Corollary 2 (Long exact sequence of a pair)**.** *Let Σ be a simplicial complex, Σ' a subcomplexes with inclusion map ι. There is a long exact sequence on homology*

$$\begin{aligned} &\cdots \xrightarrow{(\pi_*)_2} H_2(\Sigma, \Sigma') \\ &\xrightarrow{\delta_2} H_1(\Sigma') \xrightarrow{(\iota_*)_1} H_1(\Sigma) \xrightarrow{(\pi_*)_1} H_1(\Sigma, \Sigma') \\ &\xrightarrow{\delta_1} H_0(\Sigma') \xrightarrow{(\iota_*)_0} H_0(\Sigma') \xrightarrow{(\pi_*)_0} H_0(\Sigma, \Sigma') \longrightarrow 0 \end{aligned}$$

Definition. Let V, W be vector spaces. The *direct sum* $V \oplus W$ is the vector space whose elements are ordered pairs of vectors $(v, w), v \in V, w \in W$.

Corollary 3 (Mayer-Vietoris sequence)**.** *Let Σ be a simplicial complex, A, B subcomplexes. let $\iota_{X,Y}$ be the inclusion map from a subcomplex X into the larger complex Y. There is a long exact sequence on homology*

$$\begin{aligned} &\cdots \longrightarrow H_2(\Sigma) \\ &\xrightarrow{\delta_2} H_1(A \cap B) \xrightarrow{((\iota_{A\cap B,A}, \iota_{A\cap B,B})_*)_1} H_1(A) \oplus H_1(B) \xrightarrow{((\iota_{A,\Sigma} + \iota_{B,\Sigma})_*)_1} H_1(\Sigma) \\ &\xrightarrow{\delta_1} H_0(A \cap B) \xrightarrow{((\iota_{A\cap B,A}, \iota_{A\cap B,B})_*)_0} H_0(A) \oplus H_1(B) \xrightarrow{((\iota_{A,\Sigma} + \iota_{B,\Sigma})_*)_0} H_0(\Sigma) \longrightarrow 0 \end{aligned}$$