# APPLIED TOPOLOGY LECTURE NOTES

CHAD GIUSTI

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## Homological Algebra

Now that we have a firm grasp on the objects we're going to be studying, we need a quantitative framework we can use to do the computations we will use to explore properties of those objects. As is nearly always the case in mathematics, the best tool for the job is algebra, in this case a sub-field called Homological Algebra. Here, we will focus on a version built out of vector spaces so we can rely on our understanding of linear algebra. Because computers are very good at linear algebra, this is the principal approach used in applications.

### *(Review of) Linear Algebra*

What follows is a "review" in the mathematical sense: I'm going to pretend that you have a passing familiarity with these ideas whether you are or not, and it's up to you to choose where you need to spend time. You have probably seen much of this material at some point, but most of it is unlikely to have been the focus of your experience with linear algebra; some of it might not even have appeared in your linear algebra courses despite being a selection of fundamental tools in the subject, such are the vagaries of time limitations on course design. If you have not seen or do not feel comfortable with these ideas, clearly the review is doubly necessary.

We'll start from the beginning.

**Definition.** • A *field* $(\mathbb{F}, +, \cdot)$ is a set $\mathbb{F}$ equipped with binary operations $+$, called addition, and $\cdot$, called multiplication[1]. Both addition and multiplication are commutative, associative operations with identity elements 0 and 1 respectively. All elements of $\mathbb{F}$ have additive inverses and all elements besides 0 have multiplicative inverses, and multiplication distributes over addition.

[1] As usual, we will (almost always) drop the "dot" for multiplication from our notation.

- A *vector space* $(V, +)$ over $\mathbb{F}$ is a set $V$ along with a binary operation $+$, called addition. and a map $- \cdot - : \mathbb{F} \times V \to V$. Addition is associative and commutative, has an identity element 0, and additive inverses exist for every element. The identity element of $\mathbb{F}$ acts as an identity on $V$ under scalar multiplication, scalar multiplication associates with multiplication in $\mathbb{F}$, scalar multiplication distributes over $+$, and addition in $\mathbb{F}$ distributes over scalar multiplication.

- A *subspace* $W \subseteq V$ is a subset of $V$ which is a vector space with the structure induced by that on $V$.

- Let $\mathbb{F}$ be a field and $S$ a finite set. The $\mathbb{F}$-*vector space with basis* $S$ is

$$\mathbb{F}\langle S\rangle = \{\sum_{s\in S} c_s s \mid a_s \in \mathbb{F}\}.$$

The cardinality of $S$ is the *dimension* of $\mathbb{F}\langle S\rangle$[2].

That is, $\mathbb{F}\langle S\rangle$ is the set of *formal* $\mathbb{F}$-*linear combinations* of elements of $S$. If the elements of $S$ are indexed by elements of an indexing set $A$ in some way, we will often write $\sum_{a\in A} c_a s_a$. If the elements of $A$ have an order $(a(1), a(2), \ldots a(N))$, then we can write a column vector $[c_{a(1)}\ c_{a(2)} \cdots\ c_{a(N)}]^T$.

[2] The fact that dimension is well-defined is important to us, but we will not delve into it here. Recall from linear algebra that every (finite-dimensional) vector space has a basis, and that the cardinality of that basis is well-defined. To skip ahead just slightly, if $V$ is an $\mathbb{F}$-vector space then $V \cong \mathbb{F}\langle S\rangle$ for some finite set $S$, and the cardinality of $S$ is an isomorphism invariant.

*Examples.*

- $\mathbb{R}$ is a field, $\mathbb{R}^n$ is the vector space $\mathbb{R}\langle\{e_1, e_2, \ldots e_n\}\rangle$. The collection of vectors $\{(x, 2x) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

- $\mathbb{F}_2 = \{0, 1\}$ is a field, with additive identity 0, multiplicative identity 1 and $1 + 1 = 0$.

- Let $S = \{\sigma_1, \sigma_2, \sigma_3\}$. The vector space $\mathbb{F}_2\langle S\rangle$ consists of sums of the form $c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3$, where the $c_i$ are either 0 or 1.

Maps between vector spaces should, as always, preserve the structure of the vector space.

**Definition.** Let $V, W$ be $\mathbb{F}$-vector spaces and $T : V \to W$ be a function. If $T(a_1v_1 + a_2v_2) = a_1T(v_1) + a_2T(v_2)$, $T$ is a *linear transformation*. If $T$ is a bijection, it is a *(linear) isomorphism*. The *kernel* of $T$, sometimes called the *null space*, is

$$\ker(T) = \{v \in V \mid T(v) = 0\},$$

and the *image* of $T$ is

$$\mathrm{im}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\}.$$

**Lemma 1.** *Let $T : V \to W$ be a linear transformation. Then ker$(T)$ is a subspace of $V$ and im$(T)$ is a subspace of $W$.*

*Proof.* Exercise. □

Recall that choosing a matrix $A_T$ for $T$ is the performed by choosing ordered bases for $V$ and $W$. In the case where those bases are given to us, we obtain the matrix just by checking where the basis vectors go.

Given a matrix $A_T$ for a linear transformation $T$, there are straightforward algorithms for computing the image and kernel of $T$ using *Gaussian elimination*. Recall that in Gaussian row elimination, we iteratively add a scalar multiple of one row to another, or multiply a row by a scalar, to put the matrix in *row echelon form*, where each row has as its first non-zero entry a 1 that appears to the right of the first non-zero entry above it. These initial 1s are called *pivots*.

**Lemma 2.** *Let $A_T$ be the matrix of a linear transformation $T : V \to W$ and $REF(A_T)$ the row echelon form of $T$. The columns of $A$ corresponding to those of $REF(A_T)$ in which the pivots appear are a basis for $im(T)$.*

*Example.*

Consider the following matrix of a linear transformation between $\mathbb{F}_2$ vector spaces.

$$A_T = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$REF(A_T) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the image of $T$ has basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Gaussian colmun elimination works similarly, resulting in *column echelon form*.

**Lemma 3.** *Let $A_T$ be the matrix of a linear transformation $T : V \to W$. Define a block matrix*

$$A'_T = \begin{pmatrix} & & A_T & & \\ \hline 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

*If we perform Gaussian column elimination on $A'_T$ to put only the top $A_T$ block into column echelon form, $TCEF(A'_T)$ those columns in the bottom block of $TCEF(A'_T)$ for which the corresponding columns in the top block of $TCEF(A'_T)$ are zero form a basis for ker$(T)$.*

*Example.*

Back to our matrix $A_T$ from above, we have

$$A'_T = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

AND

$$TCEF(A'_T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the kernel of $T$ has basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

It is no coincidence that there are three vectors in each of these bases.

**Theorem 4** (Rank-Nullity theorem)**.** *Let $T : V \to W$ be a linear map. Then*

$$dim(im(T)) + dim(ker(T)) = dim(V).$$

*Proof.* Let $\dim(V) = N$, $\dim(\ker(T)) = k$. Fix $\{b_1, \ldots b_k\}$ a basis for $\ker(T)$, and extend this to a basis for $V$ by adding $\beta_{k+1}, \ldots \beta_N$.

Let $v \in V$, so $v$ can be uniquely written as

$$v = c_1 b_1 + \cdots + c_k b_k + c_{k+1}\beta_{k+1} + \cdots + c_N \beta_N.$$

Now,

$$\begin{aligned} T(v) &= T(c_1 b_1 + \cdots + c_k b_k + c_{k+1}\beta_{k+1} + \cdots + c_N \beta_N) \\ &= c_1 T(b_1) + \cdots + c_k T(b_k) + c_{k+1} T(\beta_{k+1}) + \cdots + c_N T(\beta_N) \\ &= c_{k+1} T(\beta_{k+1}) + \cdots + c_N T(\beta_N). \end{aligned}$$

Thus, $\{T(B_{k+1}), \ldots T(B_N)\}$ spans $\mathrm{im}(T)$.

Now, suppose there are coefficients $c_{k+1}, \ldots c_N$ so that

$$c_{k+1} T(\beta_{k+1}) + \cdots + c_N T(\beta_N) = 0.$$

But then

$$T(c_{k+1}\beta_{k+1} + \cdots + c_N \beta_N) = 0,$$

so $c_{k+1}\beta_{k+1} + \cdots + c_N \beta_N \in \ker(T)$. But each $\beta_i \in \ker(T)^\perp$, so the only element of this form in $\ker(T)$ is the origin. Thus, the $c_i$ are all zero, so the $T(\beta_i)$ are linearly independent and thus form a basis for $\mathrm{im}(T)$. Thus, $\dim(\mathrm{im}(T)) = N - k$. □