APPLIED TOPOLOGY LECTURE NOTES

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Homological Algebra

Now that we have a firm grasp on the objects we're going to be studying, we need a quantitative framework we can use to do the computations we will use to explore properties of those objects. As is nearly always the case in mathematics, the best tool for the job is algebra, in this case a sub-field called Homological Algebra. Here, we will focus on a version built out of vector spaces so we can rely on our understanding of linear algebra. Because computers are very good at linear algebra, this is the principal approach used in applications.

Homology

Now that we have our chain-complex bridge from the world of combinatorial topological spaces to algebra, we should give it a closer look.

Recall the example $\Sigma = (\{1,2,3,4\},\{123,234\})$, for which $C_\bullet(\Sigma)$ is

$$0 \xrightarrow{0} \mathbb{F}_2\langle e_{123}, e_{234}\rangle \xrightarrow{\begin{pmatrix}1&0\\1&0\\1&1\\0&1\\0&1\end{pmatrix}} \mathbb{F}_2\langle e_{12}, e_{13}, e_{23}, e_{24}, e_{34}\rangle \xrightarrow{\begin{pmatrix}1&1&0&0&0\\1&0&1&1&0\\0&1&1&0&1\\0&0&0&1&1\end{pmatrix}} \mathbb{F}_2\langle e_1, e_2, e_3, e_4\rangle \xrightarrow{0} 0$$

We showed that this chain complex is exact at C_1. It's clear that the kernel of d_2 is trivial (the zero vector space), and since it's a chain complex, the image of d_3 must also be zero – and it is, thus the complex is also exact at C_2. What about C_0? Since the complex is exact at C_1, $\ker(d_1) = \operatorname{im}(d_2)$, which is clearly 2-dimensional. By the rank-nullity theorem,

$$\dim(\operatorname{im}(d_1)) = \dim(C_1) - \dim(\ker(d_1)) = 5 - 2 = 3.$$

However, $\ker(d_0 = 0) = C_0$, so is dimension 4 – there is a gap of one dimension, so the complex is not exact at C_0. What is the meaning of this difference?

We can (and should) think about the problem from the perspective of the topological spaces, and since only 1- and 0- simplices are directly involved in constructing C_1, C_0, d_1, and d_0, we're really thinking about a topological graph.

As we just noted, the kernel of d_0 is spanned by all of the vectors affilited with the vertices, and since we're working over $\mathbb{F}_2$, elements in this vector space are just subsets of the vertices of the graph. What does it mean to be in the image of d_1? Let's look at some examples.

$$\begin{aligned} d_1(e_{12}) &= e_2 + e_1 \\ d_1(e_{13}) &= e_3 + e_1 \\ d_1(e_{12} + e_{13}) &= (e_2 + e_1) + (e_3 + e_1) = e_2 + e_3 \\ d_1(e_{12} + e_{34}) &= (e_1 + e_2) + (e_3 + e_4) \end{aligned}$$

It looks a lot like sums of pairs of basis vectors are in the image precisely if they are endpoints on a path. The final example takes two disjoint paths and gives us the endpoints of both, but that's the natural result of linearity: if we can build two things, we can also build their sum.

This all sounds strikingly familiar. We have a bunch of vertices, and we have a list of pairs of them that are path connected in the graph. Recall our definition of $\pi_0(X)$, the set of all path components of a topological space. If we knew how to take a quotient of vector spaces, this would give us exactly the same kind of information.

Definition. Let V be a vector space and W a subspace of V. The *quotient vector space* V/W is obtained by identifying all elements of W to the 0 element. Elements of V/W are *W-cosets*, written $[v] = v + W$.

Thus, in a quotient vector space we can introduce or remove linear combinations of elements of W without changing the element: $[v] = v + W = v + w + W = [v + w]$ for every $v \in V$ and $w \in W$.

Example.

Let $V = \mathbb{R}^2$ and $W = \text{span}([1\ 2]^T)$. In V/W, we have

$$\left[\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right] = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \left\{ \begin{bmatrix} a \\ 2a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\} = \left[\begin{bmatrix} 7 \\ 13 \end{bmatrix}\right] = \left[\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right]$$

Observe that $V/W \cong \mathbb{R}^1$. *However*, it is not $\mathbb{R}^1$ as we usually define it – here, elements are cosets of vectors. Just as with sets, whenever we work with quotient vector spaces, we need to be careful to do things like making sure functions are well-defined.

Since, in a chain complex, $\text{im}(d_k + 1) \subseteq \ker(d_k)$, we just do what comes naturally.

Definition. Let $C_\bullet$ be a chain complex, $k \geq 0$. The *k-th homology group (with $\mathbb{F}_2$ coefficients) of $C_\bullet$* is

$$H_k(C_\bullet) = \frac{\ker(d_k)}{\text{im}(d_{k+1})}.$$

We abbreviate $H_k(\Sigma) = H_k(C_\bullet(\Sigma))$ and $H_k(X) = H_k(C^\Delta_\bullet(X))$ for the *simplicial* and Δ *homology groups*, respectively.

In our running example, what have we done? If the sum of two basis vectors corresponding to vertices in Σ are the endpoints of a path, say $e_1 + e_2 \in \text{im}(d_1)$, then in $H_0(\Sigma)$,

$$[e_1 + e_2] = e_1 + e_2 + \text{im}(d_1) = d_1(e_{12}) + \text{im}(d_1) = 0 + \text{im}(d_1) = [0].$$

The vector $[e_1]$, on the other hand, has for example

$$[e_1] = e_1 + \text{im}(d_1) = e_1 + d_1(e_{13}) + \text{im}(d_1) = e_3 + \text{im}(d_1) = [e_3].$$

Thus, in this vector space, every vertex is equivalent to every other vertex with which it is path connected. The one-dimensional vector space $H_0(\Sigma)$ corresponds to the single path component of the simplicial complex Σ.

It should be straightforward to extend this argument to convince yourself of the following.

Lemma 1. *Let Σ be a simplicial complex (resp. Δ complex) with ℓ path components. Then $H_0(\Sigma)$ (resp. $H_0(X)$) is an ℓ-dimensional $\mathbb{F}_2$ vector space with one basis vector corresponding to each path component of Σ (resp. X).*

Swinging back around to our example one last time, we already know the complex is exact at C_1 and C_2, thus when we compute homology we will set every vector in the kernels equivalent to zero – $H_1(\Sigma) \cong H_2(\Sigma) \cong 0$. In summary, we have

$$H_*(\Sigma) = \begin{cases} \mathbb{F}_2 & * = 0 \\ 0 & * > 0 \end{cases}$$

What would these higher homology groups capture if they weren't zero?

In Σ, there were plenty of elements in the kernel of d_1, but too many in the image of d_2, which ended up "killing off" $H_1(\Sigma)$. Let's consider a modification of the complex Σ wherein we remove the facet 234 but leave its boundary edges. This new simplicial complex is $\Sigma' = (\{1, 2, 3, 4\}, \{123, 24, 34\})$. Now we have $C_\bullet(\Sigma')$ as follows

$$0 \xrightarrow{0} \mathbb{F}_2\langle e_{123} \rangle \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} \mathbb{F}_2\langle e_{12}, e_{13}, e_{23}, e_{24}, e_{34} \rangle \xrightarrow{\begin{pmatrix} 1&1&0&0&0 \\ 1&0&1&1&0 \\ 0&1&1&0&1 \\ 0&0&0&1&1 \end{pmatrix}} \mathbb{F}_2\langle e_1, e_2, e_3, e_4 \rangle \xrightarrow{0} 0$$

Observe that nothing has changed on the right side, so we can safely conclude that $H_0(\Sigma') \cong \mathbb{F}_2$. Similarly, $\ker(d_2) = 0$ just as before, so $H_2(\Sigma') = 0$. However, the structure has changed at C_1.

Using our computation from last time, we have that

$$\ker(d_1) = \mathbb{F}_2 \left\langle \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} \right\rangle \qquad \mathrm{im}(d_2) = \mathbb{F}_2 \left\langle \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} \right\rangle$$

Thus, $H_1(\Sigma') \cong \mathbb{F}_2$, and so

$$H_*(\Sigma') = \begin{cases} \mathbb{F}_2 & * = 0,1 \\ 0 & * > 1 \end{cases}$$

Observe that the generator of $H_1(\Sigma')$ is $[[1\ 1\ 0\ 1\ 1]^T]$. This equivalence class consists of exactly two vectors:

$$\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} = e_{12} + e_{13} + e_{24} + e_{34}$$

$$\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} = e_{23} + e_{24} + e_{34}$$

The second is the sum of the vectors corresponding to the boundary of the missing simplex, 234 – this vector "picks out" the cycle that should bound the missing simplex. The first is another cycle, also surrounding the hole, though now running around the perimeter. Adding the vector from the image of d_2 had the effect of "pushing" the cycle through the simplex 123, replacing the edge 23 with the edges 12 and 13.

This generalizes to all dimensions: elements of the kernel of the differential are "closed cycles", collections of simplices whose boundaries cancel out[1], and two such elements are the same if they differ by the boundary of a simplex of higher dimension.

[1] Technically, the basis elements in the chain group corresponding to these simplices, but that's a mouthful.

Definition. Let $C_\bullet$ be a chain complex. The elements of $\ker(d_k)$ are called *k-cycles* and those of $\mathrm{im}(d_{k+1})$ are called *k-boundaries*. Two cycles that differ by a boundary are called *homologous*, and they belong to the same *homology class*. The dimension of $H_k(C_\bullet)$ is the *kth Betti number*, $\beta_k(C_\bullet)$.

Informally, the *k-th Betti number* "counts the number of non-trivial k-cycles", though this is at best an approximation since we're dealing with equivalence classes, as we will see moving forward. In our examples, we have

$$\beta_k(\Sigma) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases} \qquad \beta_k(\Sigma') = \begin{cases} 1 & k = 0, 1 \\ 0 & k > 1 \end{cases}$$