APPLIED TOPOLOGY LECTURE NOTES

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Simplicial complexes

Topological spaces are extremely general, and so working with them without constraints is difficult. Further, the application-focused reader might wonder how we can perform any computations with them – how do we put all of this open set data into a computer?

Simplicial approximations of topological spaces

Simplicial complexes are powerful both because they can be used to encode very general types of data and because they can be used to approximate an extensive range of topological spaces. Let's focus on making that last statement precise by constructing such an approximation.

Definition. Let X be a topological space, A an indexing set. If $\mathcal{U} = \{U_a\}_{a\in A}$ is a family of open (closed) sets in X so that $\bigcup_{a\in A} U_a = X$, we say $\mathcal{U}$ is an *open (closed) cover* for X. If, in addition, for each $B \subseteq A$, $\bigcap_{b\in B} U_b$ is either contractible or empty[1], $\mathcal{U}$ is a *good open (closed) cover*. If X is a subspace of a Euclidean space, then if each U_a is a convex set[2] we call $\mathcal{U}$ a *convex open (closed) cover*.

[1] In particular this includes the case where B is a singleton, so each set $U_a, a \in A$ has $U_a \simeq *$ or $U_a = \emptyset$.

[2] A *convex set* A is one for which every pair of poinst $a, b \in A$, the line segment connecting a to b lies in A.

Note that we want the sets to be either all open or all closed. Open is more common in general topology, but it will be convenient for us to have the closed version for use in applications.

Examples.

- Let $X = S^1$, U_1 a contractible open set that covers slightly more than the top semicircle, U_2 similarly for the bottom. $\mathcal{U}$ is an open cover, but not a good open cover because the intersection $U_1 \cap U_2$ is not path connected, thus $U_1 \cap U_2 \not\simeq *$.
- Let $X = S^1$, U_1, U_2 and U_3 contractible open sets that each cover slightly more than a third of the circle. This is a good cover, since each $U_a cap U_b \simeq *$ for each pair a, b and $U_1 \cap U_2 \cap U_3 = \emptyset$.
- Let $X = \mathbb{R}^2$ and $U_a = \{x \in \mathbb{R}^2 \mid a - 1 < ||x|| < a + 1\}, i = 0, 1, 2, \ldots$. This is an open cover for which almost all individual U_1 are not contractible.

- Let $X = \mathbb{R}^2$ and $U_a = B_a(0), a = 0, 1, 2, \ldots$. This, on the other hand is a good cover, and a convex cover.

Covers are a way of breaking a topological space up into simpler pieces. Often, we can perform a computation on each of the simpler pieces and then glue the answers back together on the overlaps (using a quotient construction) to get the answer on the whole space, thus avoiding a more complex computation.

There are (usually) many different ways to build a cover for a space. Many spaces of interest admit *finite covers* where $|A| < \infty$, and the existence of a finite cover can be thought of as an analog of a set being finite: your intuition will generally hold because you won't have to take limits while gluing.

We can construct a simplicial complex from a finite cover by encoding the intersection patterns of the constituent sets.

Definition. Let X be a topological space and $\mathcal{U} = \{U_a\}_{a \in A}$ a finite open (closed) cover of X. The *nerve* of $|(U)$ is the simplicial complex $N(\mathcal{U}) = (A, S)$, where

$$\sigma \in S \iff \bigcap_{a \in \sigma} U_a \neq \varnothing.$$

Let's revisit our finite cover examples above.

Examples.

- In the two-set cover of S^1, we have $N(\mathcal{U})$ given by $A = \{1, 2\}$ and $F(S) = \{12\}$.
- In the three-set cover of S^1, $N(\mathcal{U})$ has $A = \{1, 2, 3\}$ and $F(S) = \{12, 13, 23\}\}$.

The second is built from a finite good cover of S^2. Further, the nerve of the cover is isomorphic to C_3, the three-node cycle graph, and $|N((U))| \simeq S^1$.[3]

Indeed, this is true in general.

[3] Contrast with the realization of the two-set cover of S^1 which is not a good cover.

Theorem 1 (Nerve theorem)**.** *Let X be a topological space and either i. $\mathcal{U}$ a finite, good open cover of X, or ii. $\mathcal{U}$ a finite, convex closed cover of X. Then $|N(\mathcal{U})| \simeq X$.*

Sketch of proof. The idea here is to build a continuous function $\eta_0 : |N(\mathcal{U})|_0 \to X$ by choosing an image for each vertex a in the corresponding U_a, then extending the map upward inductively through the higher skeleta, using the contractibility of intersections to ensure the appropriate extensions exist and to ensure that the function remains injective. Once we have a complete injective map $\eta : |N(\mathcal{U})| \to X$, contractibility can again be used to construct a continuous $H : I \times X \to X$ for which $H(0, x) = x$ and $H(1, x) \in \mathrm{im}(\eta)$ so that $H(t, x) = x$ for all $x \in \mathrm{im}(\eta)$. To obtain a map $\nu : X \to |N(\mathcal{U})|$, we compose $H(1, -)$ with the inverse of the injective η^{-1}. By construction, the composite $\nu \circ \eta = \mathrm{id}_{|N(\mathcal{U})|}$, and the function H is a homotopy $\eta \circ \nu \simeq \mathrm{id}_X$. □

The nerve theorem says that in topological spaces which admit such "nice" covers, the homotopy type of the space comes down to the combinatorics of a good/convex cover. If we *triangulate* a space, we are essentially building a closed, convex cover by simplices[4], we don't lose any homotopy-type information. Of course, we also need to make sure we get all of the continuous maps we expect.

[4] Vitally, requiring that two such simplices intersect only in a smaller simplex on the boundary of both.

Definition. Let Σ_1 and Σ_2 be simplicial complexes, and let $\phi : \Sigma_1 \to \Sigma_2$ a homomorphism. The *realization* of ϕ is the continuous function $|\phi| : |\Sigma_1| \to |\Sigma_2|$ given by linearly extending the map on the vertices to the interior of each simplex.

Theorem 2 (Simplicial approximation theorem)**.** *Let Σ_1 and Σ_2 be simplicial complexes, and let $f : |\Sigma_1| \to |\Sigma_2|$ be a continuous function. There is a subdivision[5] $D\Sigma_1$ of Σ_1 and a simplicial map $\widehat{f} : D\Sigma_1 \to \Sigma_2$ so that $|\widehat{f}| \simeq f$.*

[5] A *subdivision* of a simplicial complex involves breaking some of its simplices into collections of simplices. For example, adding a vertex in the middle of a triangle, along with all edges from the new vertex to the vertices of that triangle. The result is a "more granular" complex with the same realization – $|D\Sigma| \simeq |\Sigma|$.

Together, these results say that it is entirely reasonable (in most circumstances) to work with simplicial complexes rather than general topological spaces.