APPLIED TOPOLOGY LECTURE NOTES

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Simplicial complexes

Topological spaces are extremely general, and so working with them without constraints is difficult. Further, the application-focused reader might wonder how we can perform any computations with them – how do we put all of this open set data into a computer?

Definitions and terminology

One of the foundational results in algebraic topology is the fact that we can approximate any sufficiently nice space with a combinatorial object called a simplicial complex, which is a dramatic generalization of a graph which encodes relations that inherently involve multiple objects.

Definition. An *(abstract) simplical complex* Σ is a pair of sets $\Sigma = (V, S)$ of *vertices* and *simplices*, with $S \subseteq 2^V$ so that i. if $\sigma \in S$, $|\sigma| < \infty$; and, ii. if $\sigma \in S$ and $\tau \subset \sigma$ then $\tau \in S$.

This last property is called *subset closure*[1], and implies that we only need to specify faces which are maximal under inclusion; e.g., if $\tau \subsetneq \sigma$ and $\sigma \in S$, we can omit mention of τ when describing Σ. This dramatically reduces the amount of information needed to specify a simplicial complex. We call such maximal simplices the *facets* of the complex. As shorthand, we will usually write the set of facets, $F(S)$ in place of S when specifying Σ.

[1] This closure condition is essentially true for graphs, too: if an edge is contained in the graph, it is necessarily the case that the vertices of that edge are in the graph as well. We just don't add the vertices (or the empty set) to the list of edges.

Example.

$V =$ students, $S =$"have shared a class"

Facets of this a complex are maximal collections of students who have shared a clsss – that is, the entire class roster. Note that if we encode this information in a graph using the given relation, it is impossible to tell if three students shared a single class or each of the three pairs shared distinct classes[2].

[2] While we can work around this limitiation by enriching the graph structure, it is reaonable instead to work with data structures that naturally support the desired information (and many of the solutions people find turn out to be equivalent to moving to simplicial complexes and reinventing the wheel!)

Simplices $\sigma \in S$ with $|\sigma| = (k+1)$ are called k-simplices, written $S_k \subseteq S$. This unintuitive naming convention comes from fact that $(k+1)$ points generically span a k-dimensional space, and we are planning to make a leap to geometry shortly[3]. The k-skeleton of Σ is the subcomplex consisting of all ℓ-simplices for $\ell \leq k$.

[3] Thus, by convention, the empty simplex is a (-1)-simplex.

Definition. Let $\Sigma_1 = (V_1, S_1), \Sigma_2 = (V_2, S_2)$ be simplicial complexes. A *(simplicial complex) homomorphism* $\Phi : \Sigma_1 \to \Sigma_2$ is a function $\phi : V_1 \to V_2$ which induces a function on k-simplices $\widetilde{\phi}_k : (S_1)_k \to (S_2)_k$ for each k.

As with graphs, unless otherwise noted we will assume all simplicai complexes have $|V| = n < \infty$. In order to be explicit about constructions, it will be convenient to choose an ordering on them[4]. Thus, we will assume from here forward that $V = (1, 2, \ldots, n) = [n]$.

[4] Selecting this order is called choosing a *global orientation* and is analogous to choosing a left- or right-handed coordinate system when doing calculus in $\mathbb{R}^3$: potentially, it changes some quantities up to a sign, but these signs fall out in the end so long as everything is consistent. Since we're planning to work in $\mathbb{F}_2$ where signs don't exist, we can just ignore the issue entirely.

Using this ordering, $\sigma \in S_k$ can be written uniquely as an ordered $(k+1)$-tuple $\sigma = (i(0), i(1), \ldots i(k))$ with $i(0) < i(1) < \cdots < i(k)$. This is unwieldy, so we will denote this simplex by $\sigma_{i(0)i(1)\cdots i(k)}$ or simply $i(0)i(1)\cdots i(k)$. For example, the 3-simplex $(2, 3, 5, 8)$ is denoted σ_{2358} or 2358.[5]

[5] If this notation is ambiguous – say 23 and 58 are vertices – add delimiters as necessary.

Recall from our discussion of the standard k-simplex Δ^k that the boundary of Δ^k is made up of standard $(k-1)$-simplices. In an abstract simplicial complex, the subset closure property ensures that all subsets, called *faces*, of each simplex are contained in the complex. In particular, those faces obtained by omitting a single vertex are present.

Definition. Let $\Sigma = (V, S)$ a simplicial complex and $\sigma = i(0)\cdots i(k) \in S_k$. The *boundary* of σ is

$$\partial(\sigma) = \{i(0)\cdots\widehat{i(\ell)}\cdots i(k) \mid \ell = 0, \ldots, k\} \subseteq S_{k-1},$$

where $\widehat{i(\ell)}$ denotes omitting an index.

For example, $\partial(2358) = \{358, 258, 238, 235\}$.

With our orientation and the standard[6] ordered basis $e_0, \ldots e_k$ on $\mathbb{R}^{k+1}$, there is a canonical way to associate a k-simplex $\sigma = i(0)i(1)\cdots i(k)$ with the standard k-simplex Δ^k: send the vertex $i(\ell)$ to the unit basis vector $e_\ell \in \mathbb{R}^{k+1}$. Write Δ^k_σ for this associated space.

[6] Standard, but indexed in this non-standard way to make life simpler.

Now, if $\tau \subset \sigma \in S$, τ consists of some subset $\{i(\ell_1), i(\ell_2), \ldots i(\ell_m)\}$ of the vertices of σ. Thus, the standard simplex Δ^m_τ is canonically a subspace of Δ^k_σ given by the non-negative span of $\{e_{\ell_1}, \ldots e_{\ell_m}\}$, where the inclusion $\iota_{\tau,\sigma}$ is the linear map which takes each unit vector in Δ^m_τ to its associated unit vector in Δ^k_σ.

For example, Δ^2_{258} includes as the subspace of Δ^3_{2358} spanned by e_0, e_2, and e_3, via the linear map

$$\iota_{258,2358} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition. Let $\Sigma = (V, S)$ be a simplicial complex. The *(topological)* *realization* of Σ is the topological space given by

$$\left(\coprod_k \coprod_{\sigma \in S_k} \Delta^k_\sigma \right) / \sim$$

where if $\tau \in \partial(\sigma)$ and $x \in \Delta^{k-1}_\tau$, then $\iota_{\tau,\sigma}(x) \sim x$.