APPLIED TOPOLOGY LECTURE NOTES

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Topological spaces and continuous functions

OUR GOAL IS TO BUILD a general framework for understanding and comparing the intrinsic structure of objects. We will rely on intuition from Euclidean space as a roadmap.

Homotopy equivalence

Now that we know what our fundamental objects of study are, we need a way of deciding which topological space we're looking at when we find one in the wild – that is, we need a notion of "sameness." There are several to choose from, but we will focus on a weak notion of sameness that plays well with the tools we're going to see. First, we need a new way to build topological spaces.

Definition. Let X and Y be topological spaces. The *product* space $X \times Y$ is the topological space on the set $X \times Y$ with topology generated by the collection $U \times V$, where $U \in \tau_X$ and $V \in \tau_Y$.[1]

[1] This topology works for products of finitely many spaces. If you want infinite products, you need to refine the definition a bit.

The open sets on $X \times Y$ are precisely the minimal ones needed to make the *projection* maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ continuous.

Definition. Let X and Y be topological spaces and $f, g : X \to Y$ be continuous functions. We say f is *homotopic* to g, written $f \simeq g$, if there is a continuous function $H : I \times X \to Y$ with $H(0, x) = f(x)$ and $H(1, x) = g(x)$.

The right intuition for homotopy is that there is a "continuous path through continuous functions" from f to g.

Examples.

- Let $\iota : S^2 \hookrightarrow \mathbb{R}^3$ be the standard inclusion $\iota((x, y, z)) = (x, y, z)$ and $z : S^2 \to \mathbb{R}^3$ be the constant map $z(x, y, z) = (0, 0, 0)$. Take $H : I \times S^2 \to \mathbb{R}^3$ to be $H(t, (x, y, z)) = t(x, y, z)$, so $H(0, (x, y, z)) = z(x, y, z)$ and $H(1, (x, y, z)) = \iota$. If a function if homotopic to a constant map, we call it *null-homotopic* and write $\iota \simeq *$.
- Let $e : S^1 \hookrightarrow S^2$ by $e(x, y) = (x, y, 0)$ and $p(x, y) = (0, 0, 1)$. Let $H : I \times S^1 \to S^2$ by $H(t, (x, y)) = (x\sqrt{1 - t^2}, y\sqrt{1 - t^2}, , t)$. This is a null-homotopy of e.

- Let $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq ||(x,y)|| \leq 2\}$, take $\mathrm{id}_A, r : A \to A$ by $\mathrm{id}_A(x,y) = (x,y)$ and $r(x,y) = \frac{(x,y)}{||(x,y)||}$. We have $\mathrm{id}_A \simeq r$ via $H(t,(x,y)) = \frac{(x,y)}{(1-t)+t||(x,y)||}$.

In the last example, we deformed the identify function on an annulus to a projection onto the inner boundary circle. If we "run the homotopy backwards", we get an expansion of the circle into the whole annulus, so it seems like the annulus and the circle are "the same" in terms of continuous deformation. Let's break that up and look at the component pieces.

$$\mathrm{id}_A \circlearrowleft A \underset{\iota}{\overset{r}{\rightleftarrows}} S^1 \circlearrowright \mathrm{id}_{S^1}$$

If we take the entire annulus and pass it through r, we are collapsing it onto the circle. Then, we can send it back through the canonical inclusion to get the inner boundary of the annulus as a subspace of the annulus. Finally, we use the homotopy H to re-expand to the identiy map on the annulus. Thus, $\iota \circ r \simeq \mathrm{id}_A$. This says, "Passing A *through* S^1 via $\iota \circ r$ is the same as leaving it be, up to continuous deformation." So, S^1 is "big enough" to hold all of the information in A, up to homotopy. Observe that the same is true if we invert the roles of A and S^1: $r \circ \iota \simeq \mathrm{id}_{S^1}$ (indeed, $r \circ \iota = \mathrm{id}_{S^1}$.) Thus, each space is "at least as big" as the other, and so we should think of them as equivalent.

Definition. Let X, Y be topological spaces, $f : X \to Y$ and $g : Y \to X$ be continuous functions. If $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, we say X and Y are *homotopy equivalent* and write $X \simeq Y$. If a space is homotopy equivalent to a point, it is called *contractible*.

Examples.

- $\mathbb{R}^d \simeq *$ via $\iota : \{0\} \to \mathbb{R}^d$ by $\iota(0) = 0$ and $p : \mathbb{R}^d \to \{0\}$ the constant map.

- $\mathbb{R}^d \setminus \{0\} \simeq S^{d-1}$ via $\iota : S^{d-1} \hookrightarrow \mathbb{R}^d \setminus \{0\}$ the standard inclusion and $\nu : \mathbb{R}^d \setminus \{0\} \to S^{d-1}$ by $\nu(x) = x/||x||$.

In general, to show two things are homotopy equivalent, we must demonstrate the appropriate maps and homotopies. To show two things are *not* homotopy equivalent, we find a property of the spaces that is a *homotopy invariant* and show that that property is different between the two spaces.