APPLIED TOPOLOGY LECTURE NOTES

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Topological spaces and continuous functions

OUR GOAL IS TO BUILD a general framework for understanding and comparing the intrinsic structure of objects. We will rely on intuition from Euclidean space as a roadmap.

Topological spaces

In $\mathbb{R}^d$, we (usually) measure distance using the *Euclidean metric*[1]. Points x and y are "close together" if the distance between them is small. How small? That depends on a notion of *scale*, which depends on context. As mathematicians, we like to use $\epsilon > 0$, so y is *ϵ-close* to x if $d(x,y) < \epsilon$ To conform with our plan to work with non-quantitative structures, though, what we really want is a check we can make without looking at distance directly.

[1] What is to follow will work in any metric space, though we don't have time to get into the details.

Definition. Let $\vec{x} \in \mathbb{R}^d$ and $\epsilon > 0$. The *open ϵ-ball around x* is

$$B_\epsilon(x) = \{y \mid d(x,y) < \epsilon\}.$$

Now, if someone hands us the collection of ϵ-balls around points we can ask, "Is $y \in B_\epsilon(x)$?" as a proxy for closeness without directly computing distance.

Another useful notion from Euclidean space is that of an *open set*.

Definition. Let $U \subseteq \mathbb{R}^d$. If for every $x \in U$ there is some $\epsilon(x) > 0$ with $B_{\epsilon(x)}(x) \subseteq U$, we call U an *open set*.

Thus, if we make an $\epsilon(x)$-small measurement error when attempting to select a point x from U, we still get a point from U – membership in such sets can be measured in a relatively robust manner. This is useful in a world of imprecise measurements and noise. Notice, we can flip this characterization around: an open set is one which is a union of open balls:

$$U = \bigcup_{x \in U} B_{\epsilon(x)}(x).$$

In particular, this is true *for open balls* – they are themselves open sets!

In our setting, we won't have all of these ϵs lying around, but we're going to mimic this notion.

Definition. Let X be a set. A *topology on* X is a collection $\tau \subseteq 2^X$ of *open sets* so that

i. $X \in \tau$ and $\varnothing \in \tau$,

ii. If $U_i \in \tau$ for all i in some index set A, then $\left(\bigcup_{i \in A} U_i\right) \in \tau$, and

iii. If $U_i \in \tau$ for all $i \in A$, $|A| < \infty$, then $\left(\bigcap_{i \in A} U_i\right) \in \tau$.[2]

A *topological space* is a pair (X, τ) where X is a set and τ is a topology on X. If $U \subseteq X$ has $U \in \tau$, we say U is *open in* X. Given a point $x \in X$, every open set containing x is an *open neighborhood of* x. A *closed set* is the complement of an open set; i.e. $C \subseteq X$ is closed if there is $U \in \tau$ so that $C = X \setminus U$.[3]

[2] We'd like to have arbitrary intersections of open sets, but sadly this won't work. Consider, for example, $\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$, which is not an open set.

[3] Bad news! We have that $\varnothing$ is an open set, but X is open, so its complement, $\varnothing$, must be closed. Open and closed are *not opposites*!

Topologies are very complex objects, so we usually build them by specifying simpler families of sets that *generate* them, like our ϵ-balls from $\mathbb{R}^d$.

Definition. Let X be a set. If $\mathcal{B}$ is a collection of subsets of X, containing $\varnothing$ and X, so that every finite intersection of elements of $\mathcal{B}$ can be written as a union of elements of $\mathcal{B}$, then $\mathcal{B}$ is a *base* and *generates* some topology τ on X.

Examples.

- $\mathbb{R}^d$
  a. $\{B_\epsilon(x) \mid x \in X, \epsilon > 0\}$ is a base which generates the *standard* or *Euclidean topology* on X.
  b. $\{B_\epsilon(x) \mid x \in X, \epsilon > 0, \epsilon \in \mathbb{Q}\}$ also generates the standard topology.
  c. So does $\{B_\epsilon(x) \mid x \in \mathbb{Q}^k, \epsilon > 0, \epsilon \in \mathbb{Q}\}$
- X a set, $\tau = \{\varnothing, X\}$, the *trivial topology*.
- X a set, $\tau = 2^X$, the *discrete topology*.
- $X = \{p, q\}$, $\tau = \{\varnothing, \{p\}, \{p, q\}\}$, called the *Sierpinski space*

While it is occasionally convenient to describe a topological space from the ground up, it is usually conceptually cleaner and easier to understand a description if we describe it in relation to a space we already know.

Definition. Let (X, τ) be a topological space and $Y \subseteq X$. The subspace topology on Y is given by $\{U \cap Y \mid U \in \tau\}$.

Examples.

- Let $I = [0, 1] \subset \mathbb{R}$, the *standard closed interval*. What are the open sets in the subspace topology?

- The *standard d-sphere* is

$$S^d = \{x \in \mathbb{R}^{d+1} \mid ||x|| = 1\}$$

Open sets in S^d under the subspace topology are given by intersections of Euclidean open sets with the sphere.

- The *standard d-disk* is

$$D^d = \{x \in \mathbb{R}^d \mid ||x|| \leq 1\}$$

Observe that there is a canonical (identity) map $S^d \hookrightarrow D^{d+1}$. This is going to be one of our favorite maps.

- The *standard d-simplex* is the set

$$\Delta^d = \{x \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} x_i = 1, x_i \geq 0, i = 1, \ldots, d+1\}.$$

We should actually be thinking of the d-simplex as a subspace of the hyperplane $\sum_{i=1}^{d+1} x_i = 1$ which is itself a subspace of $\mathbb{R}^{d+1.}$