Math 829: Algebraic Topology
Prelim Exercises 2
Due Sunday, February 25th

1. Let S be a set. Define the *free group on S* to be the group with generators S and no relations. Show that
$$\mathrm{Free} : \underline{\mathrm{SET}} \to \underline{\mathrm{GRP}}$$
which takes a set S to the free group on S is a functor from the category of sets to the category of groups.

2. Let S be a set. A *word* in S is an formal expression
$$a_1 a_2 \dots a_k, \qquad k \geq 0$$
where for each i, there is some $s \in S$ with $a_i = s$ or $a_i = s^{-1}$. Write 1 for the empty word ($k = 0$). We can *reduce* a word by removing a pair $a_i a_{i+1}$ where for some $s \in S$, $a_i = s$ and $a_{i+1} = s^{-1}$ or vice versa, and *expand* a word by inserting such a pair at any position. If we cannot reduce a word, it is called *reduced*. Two words are *equivalent* if there is a sequence of expansions and reductions which takes one to the other. We say a word in S *represents* an element $g \in \mathrm{Free}(S)$ if g can be obtained by multiplying the group generators and inverses corresponding to the symbols in the word in the prescribed order. Convince yourself of the following fundamental fact, but don't get caught up proving it in detail if you don't want to.

Theorem 1. *Let S be a set. Then every element in Free(S) is uniquely represented by a reduced word in S, and two words represent the same element in F if and only if they are equivalent.*

3. What familiar group is $\mathrm{Free}(\{a\})$? Characterize all subgroups of $\mathrm{Free}(\{a\})$.

4. Give three examples of subgroups $H_1, H_2, H_3 < \mathrm{Free}(\{a, b\})$ with $H_i \cong \mathrm{Free}(\{a\})$, $i = 1, 2, 3$.

5. Give an example of a proper subgroup $H < \mathrm{Free}(\{a, b\})$ so that $h \cong \mathrm{Free}(\{a, b\})$. Is the inclusion homomorphism $\iota : H \to \mathrm{Free}(\{a, b\})$ in the image of the functor Free?

6. For which n is there a subgroup $H_n < \mathrm{Free}(\{a, b\})$ with $H_n \cong \mathrm{Free}(\{a_1, a_2, \dots a_n\})$?