Math 456: Topology and Its Applications
Homework 4
Due Friday, September 29th

1. Let Σ be the simplicial complex with vertices $V = \{1, 2, 3, 4, 5\}$ and *facets*, or maximal simplices, $F(S) = \{123, 124, 2345\}$.
   (a) List all simplices in Σ.
   (b) For each facet in Σ, compute its boundary.
   (c) For each simplex τ in the boundary of 124, write down the matrix of the linear map $\iota_{\tau,124} : \Delta^1_\tau \to \Delta^2_{124}$. Describe this visually by drawing Δ^1_τ for one choice of τ and Δ^2_{124}, and illustrating where $(1/4, 3/4)_\tau$ and $(2/3, 1/3)_\tau$ are mapped.
   (d) Draw the topological realization of Σ and illustrate the types of open sets that appear. (This might be easier with more than one picture/perspective.)

2. The following is a very useful way to build a simplicial complex out of a graph.

   Definition. Let $\Gamma = (V, E)$ be a graph. A *k-clique* in Γ is a subgraph of Γ isomorphic to the complete graph K_k. The *clique complex* of Γ is the simplicial complex $X(\Gamma)$ with vertices V and facets given by cliques of Γ.

   Let $\Gamma = (\{p, q, r, s, t\}, \{pq, pr, ps, pt, qr, qs, qt, rt, st\})$, and $\Sigma' = \Sigma$ from problem 1 with the addition of the face 134. Show that $X(\Gamma) \cong \Sigma'$ (as simplicial complexes, though this should be clear from context).

3. Consider the 2-sphere S^2. (It may be very useful here to use the result of problem 4 on homework 3.)
   (a) Write down a good, finite, open cover $\mathcal{U}$ for S^2 using exactly four open sets. (You can draw it or write down set notation – in the former case, please be very clear in your picture.) Verify that the cover is good.
   (b) Write down the simplicial complex $N(\mathcal{U})$.
   (c) Draw the realization $|N(\mathcal{U})|$. Compare/contrast to the nerve for our good, open cover of S^1 from class.
   (d) Conjecture the number of sets necessary for a good, finite open cover of S^k and what the nerve of that cover should look like.

4. Consider the simplicial complex

   $$\Sigma = (\{1, 2, 3, 4, 5, 6\}, \{123, 125, 134, 145, 236, 256, 346, 456.\})$$

   Take a closed, convex cover of $|\Sigma|$ by the realizations of the facets; for example, let U_1 be the image of Δ^2_{123} under the projection map in the quotient construction.
   (a) Draw $|\Sigma|$. What familiar object is this?

(b) Write down $N(\mathcal{U})$.

(c) Draw $|N(\mathcal{U})|$. What familiar object is this (up to a small homotopy that squishes the 3-simplexes flat)?

(d) Take the closed cover $\mathcal{V}$ of $|N(\mathcal{U})|$ by realizations of the facets, write down its nerve and draw the realization. What do you observe?

(e) Do this same process using the complex from 3(b). What happens here?

5. Write down a Δ-complex construction of the sphere S^2 using three 0-cells, three 1-cells and two 2-cells which is different than the one given in class.

6. Write down a Δ-complex which is homotopy equivalent to the torus T^2. (Hint: you can build a torus by gluing together opposite sides of a rectangle.)