Math 456: Topology and Its Applications
Homework 1
Due Friday, September 8th

1. Recall that the *line graph of order* n is a graph

$$L_n = (\{v_1, v_2, \ldots, v_n\}, \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}.$$

For $n > 2$, a *cycle graph of order* n is a graph of the form

$$C_n = (\{v_1, v_2, \ldots, v_n\}, \{v_1v_2, v_1v_n, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n\}).$$

   (a) For what pairs (n, m) are there homomorphisms $L_n \to C_m$? If there is a homomorphism, how many are there? Explain why. Be careful with these – they're subtler than they look.

   (b) Same question, $C_n \to L_m$?

   (c) Same question, $C_n \to C_m$?

   (d) Not required, but fun: same question, $L_n \to L_m$?

2. The *degree* of a vertex v in a graph $\Gamma = (V, E)$ is given by

$$||v|| = |\{e \in E \mid v \in e\}|.$$

   (a) Give an example of a graph homomorphism $\Phi : \Gamma_1 \to \Gamma_2$ so that for some vertex v, $||v|| > ||\phi(v)||$.

   (b) Similarly, find an example where $||v|| < ||\phi(v)||$.

   (c) Show that if $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ with $\Phi : \Gamma_1 \to \Gamma_2$ an isomorphism, then for each $v \in V_1$, $||v|| = ||\phi(v)||$.

3. An *automorphism* of a graph Γ is an isomorphism of the graph to itself, $\Phi : \Gamma \to \Gamma$.

   (a) Enumerate all automorphisms and justfiy your enumeration for:

      i. The line graph L_n.

      ii. The cycle graph C_n.

      iii. The graph $S_n = (\{v_0, v_1, \ldots, v_n\}, \{v_0v_1, v_0v_2, \ldots, v_0v_n\})$.

   (b) Construct a graph with $|V| > 1$ and no automorphisms besides the *trivial* automorphism that sends every vertex to itself. Prove that your graph satisfies this requirement.

4. Prove the following lemma from the lecture notes:

Definition. Let $\Gamma = (V, E)$ be a graph. Let $V' \subseteq V$ and $E' \subseteq E \bigcap \binom{V'}{2}$. The graph $\Gamma' = (V', E')$ is called a *subgraph* of Γ. If $E' = E \bigcap \binom{V'}{2}$, we say Γ' is the subgraph of Γ *induced* by V'.

Lemma. *Let* Γ_1 *and* Γ_2 *be graphs with path components* $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ *and* $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ *respectively. If* $\Gamma_1 \cong \Gamma_2$*, then* $n = m$*. Further, there is a bijection* $\psi : \mathcal{C} \to \mathcal{D}$ *so that, for each* $\imath = 1, \ldots, n$*, the subgraph of* Γ_1 *induced by* C_i *is isomorphic to the subgraph of* Γ_2 *induced by* $\psi(C_i)$*.*

5. Prove the following lemma from the lecture notes.

   Lemma. *Let* Γ *be a graph. If two vertices in* Γ *are connected by a path* π*, there is a simple path containing only edges from* π *which connects them.*

6. Find a graph Γ and two distinct planar realizations ρ_1 and ρ_2 of Γ so that you can't continuously deform ρ_1 to look like ρ_2 without causing an intersection in the vertices or edges, or of an edge and a vertex away from an endpoint. That is, there's no "path" through planar realizations between ρ_1 and ρ_2. (This one's informal – we don't have the language to make it formal yet, so no need to try to prove anthing. Just convince yourself you've got it. The best way to check is to get someone else to try.)