APPLIED TOPOLOGY LECTURE NOTES

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Homological Algebra

Now that we have a firm grasp on the objects we're going to be studying, we need a quantitative framework we can use to do the computations we will use to explore properties of those objects. As is nearly always the case in mathematics, the best tool for the job is algebra, in this case a sub-field called Homological Algebra. Here, we will focus on a version built out of vector spaces so we can rely on our understanding of linear algebra. Because computers are very good at linear algebra, this is the principal approach used in applications.

Maps on Homology

As usual, the object of study is only useful to us in the context of how objects interact. We've encoded our simplicial complexes as chain complexes, then used the differentials induced by the maps to compute a new sequence of quotient vector spaces called homology groups. What we haven't covered is what happens to homomorphisms between simplicial complexes, or more generally to continuous functions between topological spaces.

Definition. Let $C_\bullet$ and $C'_\bullet$ be chain complexes. A *(chain complex) homomorphism* $f : C_\bullet \to C'_\bullet$ is a collection of linear transformations $f_k : C_k \to C'_k$ so that $d'_k \circ f_k = f_{k-1} \circ d_k$.

That is, in the following diagram, the *squares commute* – each path around a square produces the same answer.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{k+2}} & C_{k+1} & \xrightarrow{d_{k+1}} & C_k & \xrightarrow{d_k} & C_{k-1} & \xrightarrow{d_{k-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}} & & \\
\cdots & \xrightarrow{d'_{k+2}} & C'_{k+1} & \xrightarrow{d'_{k+1}} & C'_k & \xrightarrow{d'_k} & C'_{k-1} & \xrightarrow{d'_{k-1}} & \cdots
\end{array}$$

Where can get chain complex homomorphisms? From simplicial complex homomorphisms, of course.

Lemma 1. *Let Σ, Σ' be simplicial complexes, $\Phi : \Sigma' \to \Sigma'$ a homomorphism. There is an induced homomorphism $\Phi_\# : C_\bullet(\Sigma) \to C_\bullet(\Sigma'$.*

Proof. We need to write down linear transformations $(\phi_\#)_k : C_k(\Sigma) \to C_k(\Sigma')$. It suffices to define the transformation on basis vectors, and there is only one sensible candidate:

$$(\phi_\#)_k(e_{i(0)i(1)\cdots i(k)}) = e_{\widetilde{\phi}_k(i(0)i(1)\cdots i(k))} = e_{\phi(i(0))\phi(i(1))\cdots\phi(i(k))}.$$

We need to check that this map commutes with the differentials. That is, that $(\phi_\#)_{k-1} \circ d_k = d_k \circ (\phi_\#)_k$. Again, it suffices to check on basis vectors, so we compute

$$\begin{aligned}
(\phi_\#)_{k-1}(d_k(e_{i(0)i(1)\cdots i(k)})) &= (\phi_\#)_{k-1}(\sum_{\ell=0}^{k} e_{i(0)\cdots\widehat{i(\ell)}\cdots i(k)}) \\
&= \sum_{\ell=0}^{k} (\phi_\#)_{k-1}(e_{i(0)\cdots\widehat{i(\ell)}\cdots i(k)}) \\
&= \sum_{\ell=0}^{k} e_{\phi(i(0))\cdots\widehat{\phi(i(\ell))}\cdots\phi(i(k))} \\
&= d_k(e_{\phi(i(0))\cdots\phi(i(k))}) \\
&= d_k(e_{\widetilde{\phi}_k(i(0)\cdots i(k))}) \\
&= d_k((\phi_\#)_k(e_{i(0)\cdots i(k)})).
\end{aligned}$$

□

Of course, all the work we put into defining homology would be less compelling if this wasn't a compatable story.

Lemma 2. *Let $C_\bullet$ and $C'_\bullet$ be chain complexes, and $f : C_\bullet \to C'_\bullet$ a homomorphism. For each k, there is an induced linear transformation $(f_k)_* : H_k(C_\bullet) \to H_k(C'_\bullet)$.*

Collectively, we write $f_* : H_*(C_\bullet) \to H_*(C'_\bullet)$.

Proof. There is only one sensible candidate for such a map:

$$(f_k)_*([\sigma]) = [f_k(\sigma)].$$

Clearly this map can be applied to any element in $H_k(C_\bullet)$. We need to show that this map has the right codomain, and that it is well-defined.

First, if $[\sigma] \in H_k(C_\bullet)$, $\sigma \in \ker(d_k)$. That is, $d_k(\sigma) = 0$, and since f_{k-1} is a linear transformation, $(f_{k-1} \circ d_k)(\sigma) = f_{k-1}(0) = 0$. Since f is a chain complex homomorphism, $d'_k(f_k(\sigma)) = f_{k-1}(d_k(\sigma)) = 0$, so $f_k(\sigma) \in \ker(d'_k)$ and $[f_k(\sigma)] \in H_k(C'_\bullet)$.

Now, suppose $[\sigma] = [\tau] \in H_k(C_\bullet)$. That is, $\tau = \sigma + d_{k+1}(\gamma)$ for some $\gamma \in C_{k+1}$. Thus,

$$\begin{aligned}
(f_k)_*([\tau]) &= [f_k(\tau)] \\
&= f_k(\tau) + \mathrm{im}(d'_{k+1}) \\
&= f_k(\sigma + d_{k+1}(\gamma)) + \mathrm{im}(d'_{k+1}) \\
&= f_k(\sigma) + f_k(d_{k+1}(\gamma)) + \mathrm{im}(d'_{k+1}) \\
&= f_k(\sigma) + d'_{k+1}(f_{k+1}(\gamma)) + \mathrm{im}(d'_{k+1}) \\
&= f_k(\sigma) + \mathrm{im}(d'_{k+1}) = [f_k(\sigma)] = (f_k)_*([\sigma]).
\end{aligned}$$

□

Putting these two lemmas together, any homomorphisms of simplicial complexes induces a map on homology.

Example.

Consider

$$\Sigma = (\{1,2,3,4\}, \{12,13,14,24,34\})$$

and

$$\Sigma' = (\{1',2',3',4'\}, \{1'2'3', 2'3'4', 1'4'\})$$

with homomorphism given on vertices by $\phi(i) = i'$. The chain complex for Σ is

$$0 \xrightarrow{0} \mathbb{F}_2\langle e_{12}, e_{13}, e_{14}, e_{24}, e_{34}\rangle \xrightarrow{\begin{pmatrix} 1&1&1&0&0\\ 1&0&0&1&0\\ 0&1&0&0&1\\ 0&0&1&1&1 \end{pmatrix}} \mathbb{F}_2\langle e_1, e_2, e_3, e_4\rangle \xrightarrow{0} 0$$

and for Σ' we have

$$0 \xrightarrow{0} \mathbb{F}_2\langle e_{1'2'3'}, e_{2'3'4'}\rangle \xrightarrow{\begin{pmatrix} 1&0\\ 1&0\\ 0&0\\ 1&1\\ 0&1\\ 0&1 \end{pmatrix}} \mathbb{F}_2\langle e_{1'2'}, e_{1'3'}, e_{1'4'}, e_{2'3'}, e_{2'4'}, e_{3'4'}\rangle \xrightarrow{\begin{pmatrix} 1&1&1&0&0&0\\ 1&0&0&1&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&1 \end{pmatrix}} \mathbb{F}_2\langle e_{1'}, e_{2'}, e_{3'}, e_{4'}\rangle \xrightarrow{0} 0$$

Aligning these, we can add in the induced chain maps.

$$\begin{array}{ccccccccc}
0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbb{F}_2\langle e_{12}, e_{13}, e_{14}, e_{24}, e_{34}\rangle & \xrightarrow{\begin{pmatrix} 1&1&1&0&0\\ 1&0&0&1&0\\ 0&1&0&0&1\\ 0&0&1&1&1 \end{pmatrix}} & \mathbb{F}_2\langle e_1, e_2, e_3, e_4\rangle & \xrightarrow{0} & 0 \\
 & & \big\downarrow 0 & & \big\downarrow \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} & & \big\downarrow \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} & & \\
0 & \xrightarrow{0} & \mathbb{F}_2\langle e_{1'2'3'}, e_{2'3'4'}\rangle & \xrightarrow{\begin{pmatrix} 1&0\\ 1&0\\ 0&0\\ 1&1\\ 0&1\\ 0&1 \end{pmatrix}} & \mathbb{F}_2\langle e_{1'2'}, e_{1'3'}, e_{1'4'}, e_{2'3'}, e_{2'4'}, e_{3'4'}\rangle & \xrightarrow{\begin{pmatrix} 1&1&1&0&0&0\\ 1&0&0&1&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&1 \end{pmatrix}} & \mathbb{F}_2\langle e_{1'}, e_{2'}, e_{3'}, e_{4'}\rangle & \xrightarrow{0} & 0
\end{array}$$

In the top complex,

$$\ker(d_1) = \mathbb{F}_2 \left\langle \begin{bmatrix} 1\\0\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0\\1 \end{bmatrix} \right\rangle \qquad \mathrm{im}(d_2) = 0$$

so

$$H_1(\Sigma) = \mathbb{F}_2 \left\langle \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \right\rangle$$

In the bottom complex,

$$\ker(d_1') = \mathbb{F}_2 \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle \qquad \operatorname{im}(d_2') = \mathbb{F}_2 \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

In the quotient, the first vector in the basis of the kernel is in the image, so is set to zero. Further, the sum of the three basis vectors for the kernel is the other basis element for the image, so their sum is equal to zero in the quotient space. Since the first vector is already zero, that means the sum of the other two are – or, over $\mathbb{F}_2$, they are equal. Thusm

$$H_1(\Sigma') = \mathbb{F}_2 \left\langle \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} \right\rangle$$

The map on homology is given by

$$\begin{aligned} (\phi_*)_1 \left(\begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} \right) &= \begin{bmatrix} (\phi_\#)_1 \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) \end{bmatrix} \\ &= \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} \end{aligned}$$

$$(\phi_*)_1\left(\left[\begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}\right]\right) = \left[\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}\right]$$

$$= \left[\begin{bmatrix}0\\1\\1\\0\\1\\0\end{bmatrix}\right] = \left[\begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix}\right]$$

In matrix form, using the ordered bases for homology given above, we have $(\phi_*)_1 = (1\ 1)$.

Even better, however, is the fact that moving from the world of simplicial complexes to the world of homology is compatible with our notion of sameness of maps.

Theorem 3. *Let Σ, Σ' be simplicial complexes, $f, g : \Sigma \to \Sigma'$ homomorphisms so that $|f| \simeq |g|$. Then $f_* = g_* : H_*(\Sigma) \to H_*(\Sigma')$.*

The same is true for Δ-complexes and general topological spaces with the appropriate notion of homology. We can apply this to conclude

Corollary 4. *Let Σ, Σ' be simplicial complexes. If $|\Sigma| \simeq |\Sigma'|$, then $H_*(\Sigma) \cong H_*(\Sigma')$.*

Proof. Exercise. □

The converse says that homology is a *homotopy invariant* – if two spaces have different homology, they cannot be homotopy equivalent.