APPLIED TOPOLOGY LECTURE NOTES

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UPDATED: OCTOBER 6, 2017

Homological Algebra

Now that we have a firm grasp on the objects we're going to be studying, we need a quantitative framework we can use to do the computations we will use to explore properties of those objects. As is nearly always the case in mathematics, the best tool for the job is algebra, in this case a sub-field called Homological Algebra. Here, we will focus on a version built out of vector spaces so we can rely on our understanding of linear algebra. Because computers are very good at linear algebra, this is the principal approach used in applications.

The Homology of Spheres

Let's write down some basic facts that will be useful when working with homology.

Lemma 1. *Let $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ be a sequence of vector spaces which is exact at V_2. Then $dim(V_2) = dim(im(f)) + dim(im(g))$.*

Proof. By the Rank-Nullity theorem, $\dim(V_2) = \dim(\text{im}(g)) + \dim(\ker(g))$, and by exactness, $\dim(\text{im}(f)) = \dim(\ker(g))$. ☐

Lemma 2. *Let $0 \to V_1 \xrightarrow{f} V_2$ and $V_3 \xrightarrow{g} V_4 \to 0$ be sequences of vector spaces exact at V_1 and V_4 respectively. The f is injective and g is surjective.*

Proof. We have that $\ker(f) = \text{im}(0) = 0$ and $\text{im}(g) = \ker(0) = V_4$. ☐

Corollary 3. *Let W_1, W_2, W_3 be vector spaces.*

- *If $0 \to W_1 \to 0$ is an exact sequence, $W_1 \cong 0$.*
- *If $0 \to W_1 \to W_2 \to 0$ is an exact sequence, $W_1 \cong W_2$.*
- *If $0 \to W_1 \xrightarrow{f} W_2 \xrightarrow{g} W_3 \to 0$ is a short exact sequence, f is injective and g is surjective, and*

$$dim(W_2) = dim(W_1) + dim(W_3).$$

Proof. Exercise. ☐

Lemma 4. *Let Σ, Σ' be simplicial complexes, then for all k,*

$$H_k(\Sigma \sqcup \Sigma') \cong H_k(\Sigma) \oplus H_k(\Sigma).$$

Proof. Observe that $C_k(\Sigma \sqcup \Sigma) = \mathbb{F}_2\langle S_k \sqcup S'_k\rangle$, and if $\sigma \in S_k$, $d^{\sqcup}_k(e_\sigma) \in \mathbb{F}_2\langle S_{k-1}\rangle \subseteq C_{k-1}(\Sigma \sqcup \Sigma')$ (similarly for $\Sigma \in S'_k$). Thus, $\mathrm{im}(d^{\sqcup}_k) \cong \mathrm{im}(d^{\sqcup}_k|_{S_k}) \oplus \mathrm{im}(d^{\sqcup}_k|_{S'_k}) \cong \mathrm{im}(d_k) \oplus \mathrm{im}(d'_k)$, the latter being a decomposition into the individual chain complexes for Σ and Σ'. This works similarly for kernels, and thus also for homology. □

Let $Y \subseteq X$ be a subspace and denote by X/Y the quotient space obtained by identifying all points in Y to a single point. (That is, y $\sim$ y′\) for each $y, y' \in Y$.

Lemma 5. *Let Σ be a simplicial complex and Σ' subcomplex, N the number of path components of Σ, and M the number of path components of Σ containing a vertex from Σ'. Then $H_0(\Sigma, \Sigma') \cong \mathbb{F}_2^{N-M}$.*

In particular, if Σ' is a nonempty, connected subcomplex, $H_0(\Sigma, \Sigma') = \mathbb{F}_2^{\beta_0(\Sigma)-1}$. Note that this means the relative homology isn't quite the same at dimension zero as the usual way of computing homology – we're off by one dimension most of the time.

Proof. By definition, $C_0(\Sigma, \Sigma') \cong C_0(\Sigma)/C_0(\Sigma')$.. Each path component C_i of Σ containing simplices from Σ' has at least one vertex $v_i \in \Sigma'$. In the quotient, the equivlance class of each $[e_{v_i}]$, which correspond to those connected components, is equal to $[0]$. □

Now, let's do some homology computations using our lemmas. Let $\mathrm{pt} = (\{v\}, \{v\})$ be the unique one-vertex simplcial complex – that is, a point. The chain complex for the point is

$$0 \to C_0(\mathrm{pt}) \to 0$$

so the homology is

$$H_*(\mathrm{pt}) \cong \begin{cases} \mathbb{F}_2 & * = 0 \\ 0 & \text{else} \end{cases}$$

Building on that, we have

$$H_*(S^0) \cong H_*(\{-1, 1\}) \cong H_*(\mathrm{pt}) \oplus H_*(\mathrm{pt}) \cong \begin{cases} \mathbb{F}_2^2 & * = 0 \\ 0 & \text{else} \end{cases}$$

Recall (from lecture and homework) that $D^k/S^{k-1} \simeq S^k$ and, since $S^{k-1} \simeq \partial(\Delta^k)$ and $\Delta^k \simeq D^k \simeq *$, $H_*(S^k) \cong H_*(\Delta^k, \partial(\Delta^k))$ so we can use the LES of a pair. Let's start with S^1.

Write

$$\begin{array}{l}
\cdots \longrightarrow H_2(\Delta^1) \xrightarrow{(\pi_*)_2} H_2(\Delta^1, \partial(\Delta^1)) \xrightarrow{\delta_2} \\
H_1(S^0) \xrightarrow{(\iota_*)_1} H_1(\Delta^1) \xrightarrow{(\pi_*)_1} H_1(\Delta^1, \partial(\Delta^1)) \xrightarrow{\delta_1} \\
H_0(S^0) \xrightarrow{(\iota_*)_0} H_0(\Delta^1) \xrightarrow{(\pi_*)_0} H_0(\Delta^1, \partial(\Delta^1)) \longrightarrow 0
\end{array}$$

and fill in the values we know

$$\begin{array}{l}
0 \xrightarrow{(\pi_*)_2} H_2(S^1) \xrightarrow{\delta_2} \\
0 \xrightarrow{(\iota_*)_1} 0 \xrightarrow{(\pi_*)_1} H_1(S^1) \xrightarrow{\delta_1} \\
\mathbb{F}_2^2 \xrightarrow{(\iota_*)_0} \mathbb{F}_2 \xrightarrow{(\pi_*)_0} 0 \longrightarrow 0
\end{array}$$

Since the first to vector spaces in each row will be zero all the way up after H_1, we have $0 \to H_k(S^1) \to 0$ for $k > 1$ and conclude that $H_k(S^1) \cong 0$ for $k > 1$. However, we are left with an interesting exact sequence:

$$0 \to H_1(S^1) \xrightarrow{\delta_1} \mathbb{F}_2^2 \xrightarrow{(\iota_*)_0} \mathbb{F}_2 \to 0$$

We can appeal to our lemma about short exact sequences above to get the answer immediately, but let's talk through the solution for practice. The rightmost map is zero, so the kernel is all of $\mathbb{F}_2$, thus by exactness, the image of $(\iota_*)_0$ is as well. By the Rank-Nullity theorem, $\dim(\ker((\iota_*)_0)) = \dim(\mathbb{F}_2^2) - \dim(\mathrm{im}((\iota_*)_0)) = 2 - 1 = 1$. But by exactness, $\ker((\iota_*)_0) = \mathrm{im}(\delta_1)$. Further, $\ker(\delta_1) = \mathrm{im}(0) = 0$, so the map is injective. Thus, we conclude that

$$H_*(S^1) \cong \begin{cases} \mathbb{F}_2 & * = 0, 1 \\ 0 & \text{else} \end{cases}$$

Now on to S^2.

$$\begin{array}{l}
\cdots \longrightarrow H_3(\Delta^2) \xrightarrow{(\pi_*)_3} H_3(\Delta^2, \partial(\Delta^2)) \xrightarrow{\delta_3} \\
H_2(S^1) \xrightarrow{(\iota_*)_2} H_2(\Delta^2) \xrightarrow{(\pi_*)_2} H_2(\Delta^2, \partial(\Delta^2)) \xrightarrow{\delta_2} \\
H_1(S^1) \xrightarrow{(\iota_*)_1} H_1(\Delta^2) \xrightarrow{(\pi_*)_1} H_1(\Delta^2, \partial(\Delta^2)) \xrightarrow{\delta_1} \\
H_0(S^1) \xrightarrow{(\iota_*)_0} H_0(\Delta^2) \xrightarrow{(\pi_*)_0} H_0(\Delta^2, \partial(\Delta^2)) \longrightarrow 0
\end{array}$$

and filling in, we have

$$\begin{array}{l}
\cdots \longrightarrow 0 \xrightarrow{(\pi_*)_3} H_3(S^2) \xrightarrow{\delta_3} \\
0 \xrightarrow{(\iota_*)_2} 0 \xrightarrow{(\pi_*)_2} H_2(S^2) \xrightarrow{\delta_2} \\
\mathbb{F}_2 \xrightarrow{(\iota_*)_1} 0 \xrightarrow{(\pi_*)_1} H_1(S^2) \xrightarrow{\delta_1} \\
\mathbb{F}_2 \xrightarrow{(\iota_*)_0} \mathbb{F}_2 \xrightarrow{(\pi_*)_0} 0 \longrightarrow 0
\end{array}$$

Again, we conclude that $H_k(S^2) \cong 0$ for $k > 2$. Now we have two interesting pieces

$$0 \to H_2(S^2) \xrightarrow{\delta_2} \mathbb{F}_2 \to 0$$
$$0 \to H_1(S^2) \xrightarrow{\delta_1} \mathbb{F}_2 \xrightarrow{(\iota_*)_0} \mathbb{F}_2 \to 0$$

Now, we'll apply our corollary to both to obtain

$$H_*(S^2) \cong \begin{cases} \mathbb{F}_2 & * = 0, 2 \\ 0 & \text{else} \end{cases}$$

Note that this decomposition is going to happen again for S^3, this time with the top sequence in dimension 3. This continues inductively forever, giving us

Theorem 6. *Let $k > 0$, then*

$$H_*(S^k) \cong \begin{cases} \mathbb{F}_2 & * = 0, k \\ 0 & \textit{else} \end{cases}$$