APPLIED TOPOLOGY LECTURE NOTES

CHAD GIUSTI

UPDATED: SEPTEMBER 27, 2017

Homological Algebra

Now that we have a firm grasp on the objects we're going to be studying, we need a quantitative framework we can use to do the computations we will use to explore properties of those objects. As is nearly always the case in mathematics, the best tool for the job is algebra, in this case a sub-field called Homological Algebra. Here, we will focus on a version built out of vector spaces so we can rely on our understanding of linear algebra. Because computers are very good at linear algebra, this is the principal approach used in applications.

Chain complexes

Using the notation above, it is straightforward to encode simplicial complexes in the world of vector spaces: we build vector spaces whose bases are the simplices.

Definition. Let $\Sigma = (V, S = \coprod_{k=0}^{N} S_k)$ be a simplicial complex. The *k-th (simplicial) chain group (over $\mathbb{F}_2$)* of Σ is $C_k(\Sigma; \mathbb{F}_2) = \mathbb{F}_2\langle S_k \rangle$. To make indexing easier, we set $C_{-1}(\Sigma) = 0$.

Since we will only be working over $\mathbb{F}_2$, we will omit it from notation, writing $C_k(\Sigma) = C_k(\Sigma; \mathbb{F}_2)$. And, to avoid any possible confusion, we write $e_\sigma \in C_k(\Sigma)$ for the basis vector corresponding to $\sigma \in S_k$. Finally, observe that the order on the vertices induces a lexicographic ordering on the simplices, so these vector spaces come with *canonical* ordered bases.

The boundary maps on the simplices give us induced linear transformations between the chain groups.

Definition. Let $\Sigma = (V, S = \coprod_{k=0}^{N} S_k)$ be a simplicial complex, $0 < k \leq N$. The *k-th differential* is the linear map $d_k : C_k(\Sigma) \to C_{k-1}(\Sigma)$ given on basis vectors by

$$d_k(e_\sigma) = \sum_{\tau \in \partial(\sigma)} e_\tau.$$

More explicitly,

$$d_k(e_{i(0)\cdots i(k)}) = \sum_{\ell=0}^{k} e_{i(0)\cdots \widehat{i(\ell)} \cdots i(k)},$$

where as usual the hat indicates which index to drop. By necessity, $d_0 : C_0(\Sigma) \to 0$ is the zero map.

Example.

Let Σ have $V = \{1, 2, 3, 4\}$ and $F(S) = \{123, 234\}$. The chain groups of Σ are

$$\begin{aligned} C_2(\Sigma) &= \mathbb{F}_2\langle\{123, 234\}\rangle \\ C_1(\Sigma) &= \mathbb{F}_2\langle\{12, 13, 23, 24, 34\}\rangle \\ C_0(\Sigma) &= \mathbb{F}_2\langle\{1, 2, 3, 4\}\rangle \end{aligned}$$

The differentials are the linear maps

$$d_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad d_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

These vector spaces and linear maps fit together in a sequence called a *chain complex*

$$C_2(\Sigma) \xrightarrow{d_2} C_1(\Sigma) \xrightarrow{d_1} C_0(\Sigma) \xrightarrow{d_9} 0 \; ,$$

which together we denote $C_\bullet(\Sigma)$. Consider the 2-simplex 123. The boundary of the 2-simplex is

$$\partial(123) = \{12, 13, 23\},$$

and the boundaries of each of these boundary simplices are, respectively

$$\begin{array}{rcccccl} \partial(12) = \{ & 1, & 2 & & \} \\ \partial(13) = \{ & 1, & & 3 & \} \\ \partial(23) = \{ & & 2, & 3 & \} \end{array}$$

That is, each simplex in the *boundary of the boundary* appears twice. On the level of the differentials over $\mathbb{F}_2$, we have

$$d_2(e_{123}) = e_{23} + e_{13} + e_{12},$$

and

$$d_1(e_{12} + e_{13} + e_{23}) = (e_2 + e_1) + (e_3 + e_1) + (e_3 + e_2) = 0$$

In fact, if we compose the linear maps by multiplying the matrices we wrote down (in the proper order), we get

$$d_1 d_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

To unpack that a little, let $c \in C_2(\Sigma)$. If $c' = d_2(c)$, then $d_1(c') = 0$. That is, for every $c' \in \mathrm{im}(d_2)$, $c' \in \ker(d_1)$, so $\mathrm{im}(d_2) \subseteq \ker(d_1)$.

In fact, this is true in general.

Lemma 1. *Let Σ be a simplicial complex, $C_k(\Sigma)$, $C_{k-1}(\Sigma)$ and $C_{k-2}(\Sigma)$ chain groups of Σ with differentials d_k and d_{k-1}. Then $im(d_k)$ is a subspace of $ker(d_{k-1})$; i.e. $d_{k-1} \circ d_k = 0$.*

Proof. Exercise. □

Definition. An $\mathbb{F}_2$ *chain complex,* $C_\bullet$, is a sequence of $\mathbb{F}_2$-vector spaces called *chain groups* $\{C_k\}_{k=-1}^N$ and linear maps $\{d_k : C_k \to C_{k-1}\}_{k=0}^N$ called *differentials* so that $d_{k-1} \circ d_k = 0$. By convention $C_{-1} = 0$ and d_0 is thus the zero map.

Technically, we also need to check that $\mathrm{im}(d_1) \subseteq \ker(d_0)$ in our example above to call $C_\bullet(\Sigma)$ a chain complex, but this is trivial: it is always the case that $\mathrm{im}(d_1) \subseteq C_0(\Sigma) = \ker(d_0 = 0)$.

The fact that $\mathrm{im}(d_2)$ is a subspace of $\ker(d_1)$ raises a new question: is it the *entire* kernel? To find out, we use our algorithm from the previous section to compute a basis for $\ker(d_1)$.

$$\left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

It's clear from the matrix of d_2 that the image has dimension 2, and since we have a two-element basis for the kernel, the two spaces must be isomorphic. However, since one is a subspace of the other, $\mathrm{im}(d_2) = \ker(d_1)$. Even better, we can see that the images of the basis vectors of $C_2(\Sigma)$ under d_2 are the same vectors given by the algorithm above – this sort of observation will be useful later. (*A priori,* we would also need to compute a basis for the image to work out its dimension.)

Definition. Let $C_\bullet$ be a chain complex. We say $C_\bullet$ is *exact* at C_k if $\mathrm{im}(d_{k+1}) = \ker(d_k)$. If $C_\bullet$ is exact at every k, we say it is *exact*.

A tremendous amount of modern mathematics is encoded by measuring the failure of chain complexes to be exact. We can apply the Rank-Nullity theorem in the case of exact chain complexes to obtain an interesting relationship between the dimensions of the chain groups.

Corollary 2. *Let $C_\bullet$ be an exact chain complex. Then*

$$\sum_{k=0}^{N}(-1)^k dim(C_k) = 0.$$

Since we're here, we should also define a chain complex for Δ-complexes.

Definition. Let X be a Δ-complex with cell indexing sets $\{S_k\}_{k=0}^N$ and boundary associations $b : S_k \times \{0, \dots k\} \to S_{k-1}$ as given in the definition. The Δ *chain complex (over* $\mathbb{F}_2$*)* for X is

$$C_\bullet^\Delta(X) = \left(\{C_k^\Delta(X)\}_{k=-1}^N, \{d_k : C_k^\Delta(X) \to C_{k-1}^\Delta(X)\}_{k=0}^N\right),$$

where

$$C_k^\Delta(X) = \mathbb{F}_2\langle S_k\rangle, \;\; C_{-1}^\Delta(X) = 0$$

with basis elements written e_σ for $\sigma \in S_k$, and

$$d_k(e_\sigma) = \sum_{i=0}^{k} e_{b(\sigma,i)}.$$