APPLIED TOPOLOGY LECTURE NOTES

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Topological spaces and continuous functions

Our goal is to build a general framework for understanding and comparing the intrinsic structure of objects. We will rely on intuition from Euclidean space as a roadmap.

Application: Reeb graphs

When we were discussing Jordan Curves, we mentioned path components in a context more general than graphs, but left the notion intuitive. We now have the technology to make a formal definition.

Definition. Let X be a topoogical space. A *path* in X is a continuous function $\gamma : I \to X$. The *endpoints* of the path are $a = \gamma(0)$ and $b = \gamma(1)$, and we say γ is a *path from a to b*. The existence of a path from a to b induces an equivalence relation $\sim$ on X[1]. Write $\pi_0(X) = X/\sim$, and call $\pi_0(X)$ the *path components* of X. Say X is *path connected* if $|\pi_0(X)| = 1$.

[1] Exercise.

Path components are preserved by continuous functions.

Lemma 1. *Let X, Y be topological spaces and $f : X \to Y$ a continuous function. There is an induced function on the path components $f_* : \pi_0(X) \to \pi_0(Y)$ given by $f_*([x]) = [f(x)]$.*

Proof. Because we are defining a function on a equivalence classes, what we need to show is that it is well-defined: if $[x] = [y] \in \pi_0(X)$ then $f_*([x]) = f_*([y])$. By definition, if [x] = [y] then there is a path $\gamma_{xy} : I \to X$. If we post-compose γ by f, we obtain a new continuous function[2] $(f \circ \gamma) : I \to X \to Y$ with $f \circ \gamma(0) = f(x)$ and $f \circ \gamma(1) = f(y)$. Thus, $[f(x)] = [f(y)]$ in $\pi_0(Y)$. □

[2] Exercise.

Another useful tool that comes up over and over in the study of topological spaces are real-valued functions on a topological space; we can use the ordering of points on the real line to "decompose" the space through the lens of the function.

Definition. Let X be a topological space, and $f : X \to \mathbb{R}$ a continuous function. For $c \in \mathbb{R}$, the *level set* for f at c is $X_c = f^{-1}(c)$[3]. The *sublevel set* for f at c is $X_{\leq c} = f^{-1}((-\infty, c])$, and the *superlevel set* is $X_{\geq c} = f^{-1}([c, \infty))$.

[3] This notation X_c isn't optimal: the level set clearly depends on the choice of f. However, it is standard in the literature, so we will establish it here and use it when f is clear from context, falling back on $f^{-1}(c)$ when there might be confusion.

Studying how these subspaces change as c varies will be a major theme in the course. For now, we'll combine level sets with components to get a tool that sees use in computational geometry and data visualization.

Observe that for $c < c'$, the sublevel set $X_{\leq c}$ is a subspace of the sublevel set $X_{\leq c'}$, and the inclusion map thus induces a map $(i_{cc'})_* : \pi_0(X_{\leq c}) \to \pi_0(X_{\leq c'})$. Similarly, we have $(i_{c'c})_* : \pi_0(X_{\geq c'}) \to \pi_0(X_{\geq c})$.

Definition. Let X be a topological space, and $f : X \to \mathbb{R}$ a continuous function. We construct an equivalence relation on X as follows: for each $c \in \mathbb{R}$ and $x, y \in f_c$, $x \sim y$ if they are in the same path component of f_c. The *Reeb graph* of f is $X/\sim$.

Some excellent news: under some mild hypotheses, the Reeb graph is a topological graph.

Lemma 2. *Suppose X is a topological space and $f : X \to \mathbb{R}$ a continuous function. If*

1. *there are $a, b \in \mathbb{R}$ so that $f^{-1}((-\infty, a]) = f^{-1}([b, \infty)) = \varnothing$;*
2. *there are finitely many points $a = c_1 < c_2 < \cdots < c_k = b \in \mathbb{R}$ so that for $i = 1, \ldots, k-1$*
   - *if $c < c' \in [c_i, c_i + 1)$, the map $(i_{cc'})_* : \pi_0(X_{\leq c}) \to \pi_0(X_{\leq c'})$ is a bijection, and*
   - *if $c < c' \in (c_i, c_i + 1]$, the map $(i_{c'c})_* : \pi_0(X_{\geq c'}) \to \pi_0(X_{\geq c})$ is a bijection; and*
3. *$|\pi_0(X_c)| < \infty$ for all $c \in \mathbb{R}$.*

Then the Reeb graph of f is a topological graph.