APPLIED TOPOLOGY LECTURE NOTES

CHAD GIUSTI

UPDATED: SEPTEMBER 9, 2017

Topological spaces and continuous functions

OUR GOAL IS TO BUILD a general framework for understanding and comparing the intrinsic structure of objects. We will rely on intuition from Euclidean space as a roadmap.

Continuous functions

Just like with graphs, we want a notion of how to compare topological spaces. The "correct" choice of maps should preserve the data of the topological space: open sets.

We might want to say that $f : X \to Y$ is continuous if it sends open sets to open sets. However, consider the constant map

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto 0.$$

Clearly this should be continuous, but it maps the open set $\mathbb{R}$ to the non-open set $\{0\}$.

Note that a function f is continuous at x if the image of a small neighborhood N_x of x, $f(N_x)$ gets arbitrarily small as N_x does. That is, if we consider a small open neighborhood of $f(x)$, it should be the image of a small open neighborhood of x.

Definition. Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if for every open set $U \subseteq Y$, the set $f^{-1}(U) \subseteq X$ is open. We call continuous functions *maps*.

In our constant function example above, if $U \subseteq \mathbb{R}$ is an open neighborhood of zero, $f^{-1}(U) = f^{-1}(0) = \mathbb{R}$ which is open. If V is an open set in $\mathbb{R}$ which doesn't contain zero, then $f^{-1}(V) = f^{-1}(\varnothing) = \varnothing$, which is also open. Thus, the function is continuous.

Theorem 1. *For a function $f : \mathbb{R} \to \mathbb{R}$, the above definition of continuity agrees with the usual ϵ-δ definition from calculus.*

Proof. Exercise. □

Our definitions of the subspace topology is motivated by the desire to make the natural inclusion of sets into a map in the simplest possible way.

Lemma 2. *Let X be a topological space, $Y \subseteq X$ equipped with the subpace topology. The* inclusion map $\iota : Y \hookrightarrow X$ *given by* $\iota(x) = x$ *is continuous.*

Proof. Let $U \subseteq X$ be open. The preimage is $\iota^{-1}(U) = \{y \in U \mid y \in Y\} = U \cap Y$, which is open in Y by definition, so ι is continuous. □

One of the most powerful tools we'll have in our quest to build new topological spaces are *quotient spaces*. Quotients provide us with the power to "glue pieces together" or "forget information", depending on context. We build quotient spaces by defining an *equivalence relation*[1] on the space X and then building the simplest topology that makes the induced *projection* map continuous.

[1] Recall that an equivalence relation on a set X is a relation $\sim$ which is reflexive, symmetric and transitive. Write $[x]$ for the *equivalence class* of x. The equivlance classes under $\sim$ partition X, so we can think of the collection of classes as a new set $(X/\sim) = \{[x] \mid x \in X\}$, the *quotient of X by* $\sim$.

Definition. Let (X, τ) be a topological space and $\sim$ an equivalence relation on X. The quotient map $\pi : X \to (X/\sim)$ is given by $\pi(x) = [x]$, and the quotient topology on $(X/\sim)$ is such that $U \in (X/\sim)$ is open if and only if the preimage $\pi^{-1}(U)$ is open.

Quotient spaces are one of the most powerful tools we have for building spaces.

Examples.

- Let $\Gamma = (V, E)$ be a graph. The *topological realization* of Γ is the topological space given by[2]

  $$|\Gamma| = \Big(\coprod_{v \in V} \Delta^0 \times \coprod_{vv' \in E} \Delta^1\Big) / \big((1,0)_{vv'} \sim v, (0,1)_{vv'} \sim v'\big).$$

  Open sets in this space inside of edges are just the same as usual. If they cross a vertex, they've got to bleed into *all* nearby edges.

[2] This is a lie. In order to do this correctly, what we really need to do is make a choice for each edge of a vertex to attach at 0 and the other to attach at 1. That requires notation that obscures a simple idea, so I didn't write it down.

With this notion of topological realization of a graph, we can redefine the earlier notion of a geometric realization in a cleaner, more understandable way.

Definition. Let Γ be a graph. A geometric realization ρ of Γ is an injective, continuous function $\rho : \Gamma \to \mathbb{R}^2$.